Stochastic solution of Cauchy problems

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"*Probability is simply a branch of measure theory, with its own special emphasis and field of application ...*"

J.L. Doob

in his book "Stochastic processes", Wiley, New York, 1953.

Probability and transforms

If the random variable Y has density $f(x)$ so that

$$P(a \leq Y \leq b) = \int_a^b f(x)dx$$

then $f(x)$ has Fourier transform

$$\begin{aligned} \hat{f}(k) = E(e^{-ikY}) &= \int_{-\infty}^{\infty} e^{-ikx} f(x)dx \\ &= \int_{-\infty}^{\infty} (1 - ikx + \frac{1}{2!}(ikx)^2 + \cdots) f(x)dx \\ &= 1 - ik\mu_1 - \frac{1}{2!}k^2\mu_2 + \cdots \end{aligned}$$

where the lth moment is $\mu_l = \int_{-\infty}^{\infty} x^l f(x)dx$

Central limit theorem

If $\mu_1 = 0$ and $\mu_2 = 2$ then $\hat{f}(k) = 1 - k^2 + \cdots$
The IID sum $S(n) = Y_1 + \cdots + Y_n$ has FT $\hat{f}(k)^n$ and the normalized sum $S(n)/\sqrt{n}$ has FT

$$\left(\hat{f}(k/\sqrt{n})\right)^n = (1 - (k/\sqrt{n})^2 + \cdots)^n$$
$$= \left(1 - \frac{k^2}{n} + \cdots\right)^n$$
$$\to e^{-k^2} \equiv \hat{g}(k) \quad \text{as} \quad n \to \infty.$$

Inverting the Fourier transform reveals a Gaussian(Normal) density

$$g(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}$$

Brownian motion

If Y_n represents a particle jump at time n then $S(n) = Y_1 + \cdots + Y_n$ is the location of the particle at time n. Expanding the time scale by a factor of $c > 0$ and taking limits as $c \to \infty$ shows that $c^{-1/2}S([ct]) \implies W(t)$ since

$$\hat{f}(c^{-1/2}k)^{[ct]} = \left(1 - \frac{k^2}{c} + \cdots\right)^{[ct]}$$
$$\to e^{-k^2 t} \equiv \hat{p}(t,k) \quad \text{as} \quad c \to \infty$$

for all $t > 0$. Inverting the FT shows that the density of the limiting Brownian motion process $W(t)$ is Gaussian (Normal)

$$p(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Brownian motion

Classical random walk

$$S(t) = Y_1 + \cdots + Y_{[t]}$$

A particle takes a random jump Y_n at time $t = n$. Particle location at time t is a simple random walk $S(t)$ and scaling limit is a Brownian motion.

$$c^{-1/2} S(ct) \Rightarrow W(t) \approx N(0, \sigma^2 t) \qquad (c \to \infty)$$

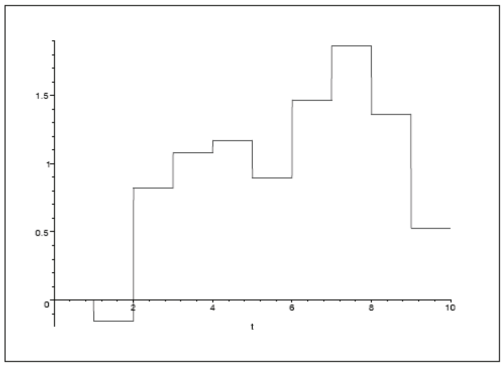
Contract spatial scale ($c^{-1/2}$)

Expand time scale (ct)

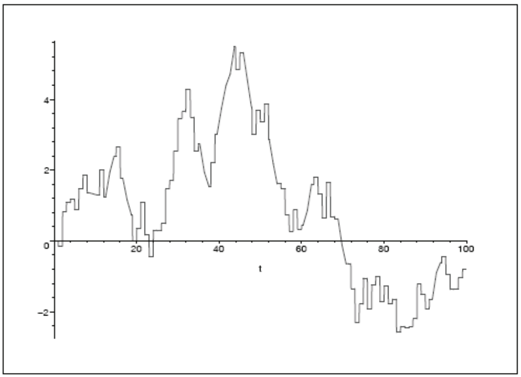
Normal limit density ($N(0, \sigma^2 t)$)

Add an advective drift: $L(t) = vt + W(t) \approx N(vt, \sigma^2 t)$

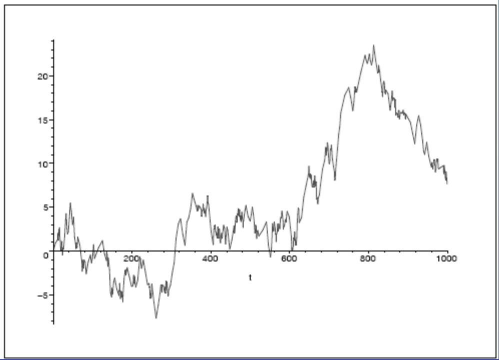
Random walk simulation



Longer time scale



Scaling limit: Brownian motion



Random graph of fractal dimension 1.5 and no jumps.

Most likely shape of a Brownian path.

Microsoft stock-the last two years

Some history of Brownian motion (BM)

- Robert Brown (1827), a Botanist: was first to observe that pollen grains in water move continuously and very erratically.
- Louis Bahcelier (1900): presented a stochastic analysis of the stock and option markets using BM
- Albert Einstein (1905): used BM to determine the law of the position of the particle...
- Norbert Wiener (1923): Mathematical foundations of BM
- Doob (1956): connections to analysis, heat equation
- Kolmogorov, Lévy, Khintchine, ...

Derivatives and transforms

- If the Laplace transform of $f(t)$ is defined for $s > 0$ by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

then $\frac{df(t)}{dt}$ has Laplace transform $s\tilde{f}(s) - f(0)$.

- If the Fourier transform of $f(x)$ is defined for $k \in \mathbb{R}$ by

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-ikx} f(x) dx$$

then $\frac{df(x)}{dx}$ has Fourier transform $ik\hat{f}(k)$.

The diffusion (heat) equation

Taking Fourier transforms in the classical diffusion equation

$$\frac{\partial p(t,x)}{\partial t} = \frac{\partial^2 p(t,x)}{\partial x^2}$$

yields

$$\frac{\partial \hat{p}(t,k)}{\partial t} = (ik)^2 \hat{p}(t,k) = -k^2 \hat{p}(t,k)$$

whose solution

$$\hat{p}(t,k) = e^{-k^2 t}$$

inverts to the same limit density for the Brownian motion $W(t)$.
For a cloud of diffusing particles $p(t,x)$ is the particle density.

Let $W_t \in \mathbb{R}$ be Brownian motion started at x. Then the function (convolution of f and $p(t,x)$)

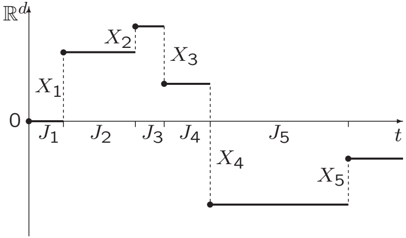
$$u(t,x) = E_x[f(W(t))] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} f(y)dy$$

solves the heat equation

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{\partial^2 u(t,x)}{\partial x^2}, \qquad t > 0, \quad x \in \mathbb{R} \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}. \end{aligned}$$

This is due to J.L. Doob (1956).
In this case we say, Brownian motion $W(t)$ is a stochastic solution of the heat equation.

Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

Heavy tailed waiting times

Random wait J_n between jumps, nth jump time given by a random walk

$$T(n) = J_1 + \cdots + J_n$$

Number of jumps by time t is inverse $N(t) \geq n \longleftrightarrow T(n) \leq t$

For heavy tail waiting times $P(J_n > t) \approx Ct^{-\beta} \quad (0 < \beta < 1)$

$$c^{-1/\beta} T(ct) \Rightarrow P(t) \quad \Longleftrightarrow \quad c^{-\beta} N(ct) \Rightarrow Q(t)$$

Inverse processes have inverse scaling

$$P(ct) \approx c^{1/\beta} P(t) \quad \Longleftrightarrow \quad Q(ct) \approx c^{\beta} Q(t)$$

Continuous time random walks (CTRW)

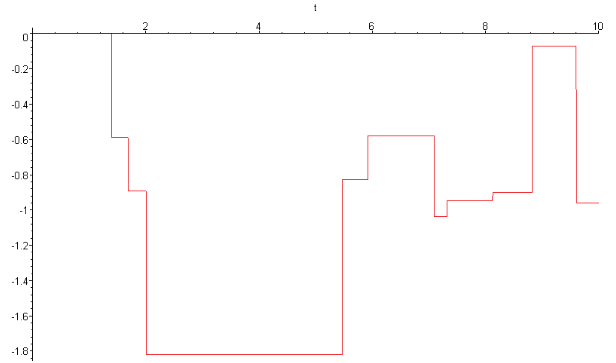
Particle jump random walk has scaling limit
$c^{-1/2}S([ct]) \implies W(t)$.
Number of jumps has scaling limit $c^{-\beta}N(ct) \implies Q(t)$.
CTRW is a random walk subordinated to (a renewal process) $N(t)$

$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}$$

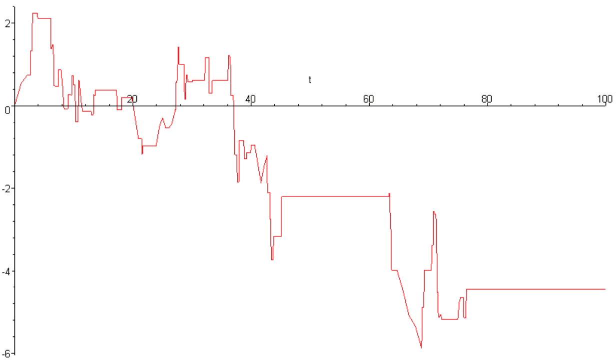
CTRW scaling limit is a subordinated process:

$$\begin{aligned} c^{-\beta/2}S(N(ct)) &= (c^{\beta})^{-1/2}S(c^{\beta}\cdot c^{-\beta}N(ct)) \\ &\approx (c^{\beta})^{-1/2}S(c^{\beta}Q(t)) \implies W(Q(t)). \end{aligned}$$

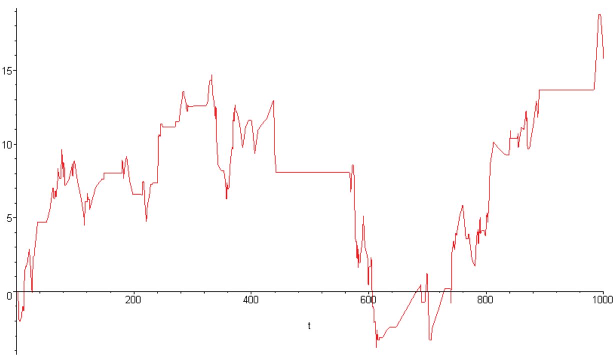
CTRW simulation with heavy tail waiting times



Longer time scale



Scaling limit: Subordinated motion



Limit retains long waiting times.


S&P 500 INDEX, RTH
15: 58 01:10: 2009
^GSPC
1070
1065
1060
1055
1050
1045
1040
1035
1030
1025
25.09
28.09
29.09
30.09
1.10
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Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)

Fractional derivatives: An old idea gets new life

- Fractional derivatives $D^\beta f(x)$ for any $\beta > 0$ were invented by Leibniz (1695) soon after the more familiar integer derivatives.
- The Caputo fractional derivative of order $0 < \beta < 1$ defined by

$$D_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^\beta} \tag{1}$$

  was invented to properly handle initial values (Caputo 1967).
- Laplace transform of $D_t^\beta g(t)$ is $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

examples

- $$D_t^\beta(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$
- $$D_t^\beta(e^{\lambda t}) = \lambda^\beta e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$
- $$D_t^\beta(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

Time-fractional model for anomalous sub-diffusion

Nigmatullin (1986), Zaslavsky (1994) studied the Cauchy problem

$$\frac{\partial^\beta}{\partial t^\beta} u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2}; \quad u(0,x) = f(x) \tag{2}$$

that models particles that wait for a time J_n before the nth jump, where $P(J_n > t) \approx t^{-\beta}$ for some $0 < \beta < 1$.

The solution to time-fractional diffusion is given by

$$u(t,x) = E_x(f(W(Q(t)))) = \int_0^\infty q(u,x) f_{Q(t)}(u) du$$

$q(u,x) = E_x(f(W(u)))$ solution to the heat equation.

$f_{Q(t)}(u)$ inverse stable density of $u = Q(t)$

In finance, $Q(t)$ represents the number of trades by time t.

Proof uses Fourier and Laplace transforms and inverting these transforms.

Taking Fourier-Laplace transform of the Equation (2) gives

$$\begin{aligned}\bar{u}(s,k) &= \frac{s^{\beta-1}\hat{f}(k)}{s^{\beta}+k^2} \\ &= s^{\beta-1}\int_0^{\infty}\exp(-[s^{\beta}+k^2]l)\hat{f}(k)dl\end{aligned} \tag{3}$$

The next step is to invert this Fourier-Laplace transform using the fact that $Q(t)$ has density

$$f_{Q(t)}(s) = \frac{t}{\beta}g_{\beta}(\frac{t}{s^{1/\beta}})s^{-1/\beta-1}, \text{ and } \int_0^{\infty} e^{-su}g_{\beta}(u) = e^{-s^{\beta}}.$$

In the case $\beta = 1/2$,

$$f_{Q(t)}(s) = \frac{2}{\sqrt{4\pi t}}e^{-s^2/4t} = f_{|W(t)|}(s)$$

This proof is due to Meerschaert, Benson, Scheffler and Baeumer (2002)

Equivalence to Higher order PDE's

- For any $m = 2, 3, 4, \ldots$ both the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} \Delta^j f(x) + \Delta^m u(t,x); \quad u(0,x) = f(x) \tag{4}$$

and the fractional Cauchy problem:

$$\frac{\partial^{1/m}}{\partial t^{1/m}} u(t,x) = \Delta u(t,x); \quad u(0,x) = f(x), \tag{5}$$

have the same unique solution given by

$$u(t,x) = \int_0^\infty p((t/s)^{1/m}, x) g_{1/m}(s)\, ds = E_x(f(W(Q(t))))$$

- Due to Baeumer, Meerschaert, and Nane TAMS(2009).

Connections to iterated Brownian motions

- Orsingher and Benghin (2004) and (2008) show that for $\beta = 1/2^n$ the solution to

$$\frac{\partial^{1/2^n}}{\partial t^{1/2^n}} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \tag{6}$$

is given by running

$$I_{n+1}(t) = W_1(|W_2(|W_3(|\cdots(W_{n+1}(t))\cdots|)|)|)$$

Where W_j's are independent Brownian motions, i.e., $u(t,x) = E_x(f(I_{n+1}(t)))$ solves (6), and solves (4) for $m = 2^n$.

Corollary

- We obtain the equivalence of one dimensional distributions in the case $Q(t)$ is the inverse stable subordinator of index $\beta = 1/2^n$

$$I_{n+1}(t) = W_1(|W_2(|W_3(|\cdots(W_{n+1}(t))\cdots|)|)|) \stackrel{(d)}{=} W_1(Q(t))$$

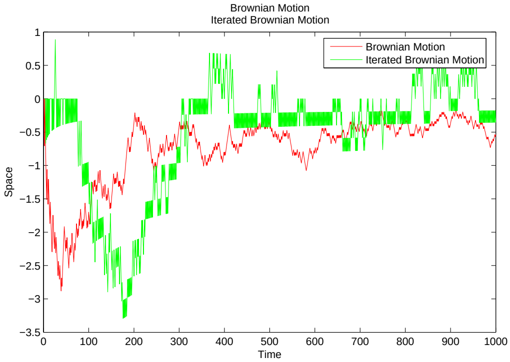

Figure: Simulations of iterated Brownian motions

Heat equation in bounded domains

Heat equation in D with Dirichlet boundary conditions:

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \Delta u(t,x), \quad x \in D, \ t > 0, \\ u(t,x) &= 0, \quad x \in \partial D; \quad u(0,x) = f(x), \quad x \in D. \end{aligned}$$

When $D = (0, M)$ can be solved by separation of variables: set $u(t,x) = \phi(x)T(t)$. Hence $\phi(x)$ satisfies

$$\frac{\partial^2 \phi(x)}{\partial x^2} = -\lambda\phi(x), \quad x \in (0, M), \lambda > 0; \quad \phi(0) = 0, \quad \phi(M) = 0$$

and $T(t)$ satisfies $T'(t) = -\lambda T(t); \quad T(0) = 1.$

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}, T_n(t) = e^{-\lambda_n t}.$$

Same applies in any dimension $d \geq 1$.

Denote the eigenvalues and the eigenfunctions of Δ_D by $\{\lambda_n, \phi_n\}_{n=1}^{\infty}$, where $\phi_n \in C^\infty(D)$. The corresponding heat kernel is given by

$$p_D(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y).$$

The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$. In this case, the semigroup given by

$$\begin{aligned} T_D(t)f(x) &= E_x[f(W(t))I(t < \tau_D(X))] = \int_D p_D(t,x,y)f(y)dy \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\bar{f}(n) \end{aligned}$$

solves the Heat equation in D with Dirichlet boundary conditions.

Fractional diffusion in bounded domains

$$\begin{aligned} \frac{\partial^\beta}{\partial t^\beta} u(t,x) &= \Delta u(t,x); \quad x \in D, \ t > 0 \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D. \end{aligned} \tag{7}$$

Separation of variables gives the unique (classical) solution as

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) M_\beta(-\lambda_n t^\beta) \\ &= E_x[f(W(Q(t)))I(\tau_D(W) > Q(t))] \\ &= E_x[f(W(Q(t)))I(\tau_D(W(Q)) > t)] \\ &= \frac{t}{\beta}\int_0^\infty T_D(l) f(x) g_\beta(t l^{-1/\beta}) l^{-1/\beta - 1} dl \end{aligned}$$

Joint work with Meerschaert and Vellaisamy, AOP (2009).

Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).

In this case, eigenfunctions and eigenvalues are

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}$$

The time fractional diffusion on $(0, M)$ has the solution

$$u(t,x) = \sum_{n=1}^{\infty} \bar{f}(n) \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M) M_\beta(-\lambda_n t^\beta)$$

here

$$M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(1+\beta n)}$$

For $\beta = 1$, $M_1(-z) = e^{-z}$, and u coincides with the solution of the heat equation on $(0, M)$.

IBM in bounded domains

The (classical) solution of

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= \frac{\Delta f(x)}{\sqrt{\pi t}} + \Delta^2 u(t,x), \quad x \in D, \ t > 0; \qquad (8) \\ u(t,x) &= \Delta u(t,x) = 0, \ t \geq 0, \ x \in \partial D; \\ u(0,x) &= f(x), \ x \in D \end{aligned}$$

is given by (running $I_2(t) = W_1(|W_2(t)|)$, IBM)

$$\begin{aligned} u(t,x) &= E_x[f(I_2(t)) I(\tau_D(W_1) > |W_2(t)|)] \\ &= 2\int_0^\infty T_D(l) f(x) h(t,l) dl, \qquad (9) \end{aligned}$$

where $T_D(l)$ is the heat semigroup in D, and $h(t,l)$ is the transition density of one-dimensional Brownian motion $\{W_2(t)\}$.
Proof: equivalence with fractional Cauchy problem for $\beta = 1/2$.

Extensions

Open problems

- Extension to Neumann boundary conditions...
- Extension to $\beta > 1$
- Extension to other operators: distributed order, Volterra-type integro-differential operators
- Work in progress for the Subordinated Brownian motions, e.g. symmetric stable process as the outer process....
- Applications-interdisciplinary research

Conclusion

- Math solves real world problems
- Probability is a useful area of Math in solving real world problems