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# An Introduction to Superprocesses

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## “The Mathematics of Life”

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# 1. Branching Brownian Motion

Consider a finite number of particles, moving erratically in $d$-dimensional space according to independent *Brownian motions*. Each particle has an exponential life length, after which it either dies or splits into two new particles, with equal probability $\frac{1}{2}$. The whole system of particles is called a *branching Brownian motion*.

The total number of particles evolves like a *Galton–Watson branching process*, which is *critical*, in the sense that the *expected* population size remains constant. More precisely, the expected number of particles at different locations evolves deterministically according to the heat equation. However, the random evolution of the whole population is given by a complicated family tree, involving both random branching and erratic motion along each branch.

## 2. Diffusion Approximation

Now let a huge number of microscopic particles perform a branching Brownian motion. Suppose we start with $n$ particles at time 0, each of mass $1/n$. We also speed up the branching by letting the average life length be $1/n$. Then as $n \to \infty$, the whole polulation, regarded as a mass distribution on $\mathbb{R}^d$, approaches *a diffuse cloud that evolves randomly in time.*

With the branching taken out, the cloud will evolve deterministically, in the same way as heat propagates through a body. (In other words, the evolution is then described by the *heat* or *diffusion equation*, which leads to a normal distribution of particles at each time.) When we add the branching, the evolution becomes truly random and unpredictable. The random cloud is then called a *superprocess*. Such processes have been studied intensely for the last 30 years.

## 3. Persistence—Extinction

If we start with a finite total mass, the process will eventually die out, although the *expected* population size remains constant. Thus, after a long time $t$, the population will most likely be extinct. This is compensated by a large population size in the rare case of survival.

To avoid extinction, we need to start with an infinite total mass, which may be taken to be uniformly spread out in space. Here the branching leads to *clumping*, whereas the spatial motion tends to *dissolve the clumps*. Which one of these tendencies will win in the long run depends on the dimension. In dimensions 1 and 2, the clumping wins and the process becomes *locally* extinct. In other words, the clumps get heavier and heavier and become separated by huge areas of empty space (like galaxies in space). If instead $d \geq 3$, the process eventually approaches a *steady-state* distribution. (The cloud keeps evolving, it is the probabilities that converge.)

# 4. Fractal Properties

While the one-dimensional superprocess has a nice density at every time $t > 0$, in dimensions $d \geq 2$ the process becomes extremely irregular. Indeed, the mass distribution is *singular*, in the sense of being supported by a set of volume 0. The size of such a set can be measured by its *Hausdorff dimension*. (For example, a surface in space has H-dimension 2, the famous *Cantor set* has H-dimension $\log 2/\log 3$, etc.) It turns out that, for fixed $t > 0$, the $d$-dimensional superprocess has H-dimension 2 for all $d \geq 2$.

Looking at a tiny part of the process under the microscope, we see a mass distribution that is both *stationary* (shift invariant in distribution) and *self-similar* (invariant under suitable scaling). Lower-dimensional sets with such invariance and scaling properties are often called *fractals*. In this sense, a superprocess at a fixed time is an example of a *random fractal*.

## 5. Family Structure

Think of the superprocess at a fixed time $t$ as an infinite (in fact uncountable) collection of infinitesimal particles. Quite surprisingly, at any earlier time $s = t - h$, only *finitely many individuals* contribute to this huge population. All other branches of the family tree have died out. (This is related to a well-known phenomenon in evolutionary biology: All humans have a common ancestor, some 100,000 generations back; similarly for all primates, all mammals, etc.)

Each of these ancestors gives rise to a *random cluster*. Clusters of different age $h$ have similar shape, apart from a scaling. (Older clusters tend to get larger.) The individual clusters have a simple description in terms of a branching Brownian motion, involving *binary splitting, but no deaths* this time. (No need to take a scaling limit.)

## 6. Brownian Snake

Superprocesses can be generated by means of a *Brownian snake*. The construction requires a *contour process* $L$, determining the length of the snake at all times. For any fixed time $s$, the snake is given by a Brownian motion in $\mathbb{R}^d$ defined on the time interval $[0, L(s)]$. When $L$ descreases, we just reduce the length of the snake; when $L$ increases, we extend the snake to its new length $L(s)$, following the path of an *independent* Brownian motion. Think of a *haunted dragon* from some myth or fairy tale: If you cut off his head, it will soon grow out again.

The point of the snake is that, if we choose the contour process to be the positive parts of a one-dimensional Brownian motion, then the head of the snake will trace out the entire superprocess, at all times. (Length of the snake corresponds to time for the superprocess.) Some of the more subtle properties of superprocesses have been deduced from a detailed study of the Brownian snake.

# 7. Evolution Equations

The expected values of the superprocess evolve according to the *heat* or *diffusion equation*

$$\dot{u} = \tfrac{1}{2}\Delta u.$$

The same equation governs the evolution of the cloud itself, only when the branching is taken out. For the actual superprocess, the evolution of the *probability distributions* (or rather their Laplace transforms) is governed by the semi-linear equation

$$\dot{u} = \tfrac{1}{2}\Delta u - u^2.$$

An analysis of this equation gives valuable information about the superprocess, and also *the other way around.* The process itself is governed by the *stochastic partial differential equation*

$$\dot{u} = \tfrac{1}{2}\Delta u + \sqrt{2u}\,\dot{w},$$

where $\dot{w}$ represents *space-time white noise.* The problem is that, for $d \geq 2$, solutions exist only in the weak (distributional) sense, if at all.

## 8. Historical Notes

The heat or diffusion equation was introduced and studied by FOURIER (1822). The physical phenomenon of Brownian motion was first observed in the microscope by VAN LEUWENHOEK (∼1674) and later studied in minute detail by a botanist named BROWN (1828). The mathematical description was proposed by EINSTEIN (1905). But then BM had already been introduced and studied by BACHELIER (1900) to model fluctuations on the stock market. The existence of BM as a mathematical object was established by WIENER (1923).

Branching processes were introduced by GALTON and WATSON (1874) to study the survival of family names. In this context, the diffusion limit was noted by FELLER (1951). Measure-valued branching processes were first considered by JIRINA (1958). Superprocesses were introduced by WATANABE (1968) and DAWSON (1977) and have been studied extensively ever since. The literature on the subject is absolutely staggering, both in volume and depth. The quest goes on.