Stochastic Dynamical Systems and SDE's

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An Informal Introduction

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or

Probability without tears

Deterministic system, discrete time

Simple recursion:

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots$$

More generally:

$$x_{n+1} = f_n(x_0, x_1, \ldots, x_n)$$

Reduction to the previous case:

$$y_n = (n, x_0, x_1, \ldots, x_n)$$

Recursion with randomness (Markov chain)

Recursion:

$$X_{n+1} = f(X_n, \xi_{n+1}), \quad n = 0, 1, \ldots$$

with $\xi_1, \xi_2, \ldots$ independent randomization variables

More generally:

$$X_{n+1} = f_n(X_0, \ldots, X_n; \xi_1, \ldots, \xi_{n+1})$$

Reduction to previous case:

$$Y_n = (n, X_0, \ldots, X_n; \xi_1, \ldots, \xi_n)$$

Recursion in terms of increments

Recursion:

$$\begin{aligned}\Delta X_n &= X_{n+1} - X_n \\ &= f(X_n, \xi_{n+1}) - X_n = g(X_n, \xi_{n+1})\end{aligned}$$

Summation:

$$X_n = X_0 + \sum_{k=0}^{n-1} g(X_k, \xi_{k+1})$$

Space-homogeneous case (random walk)

Assume $g(x, u)$ is independent of x:

$$\Delta X_n = g(\xi_{n+1}) = \eta_{n+1}$$

Here $\eta_1, \eta_2, \ldots$ are i.i.d. random variables.

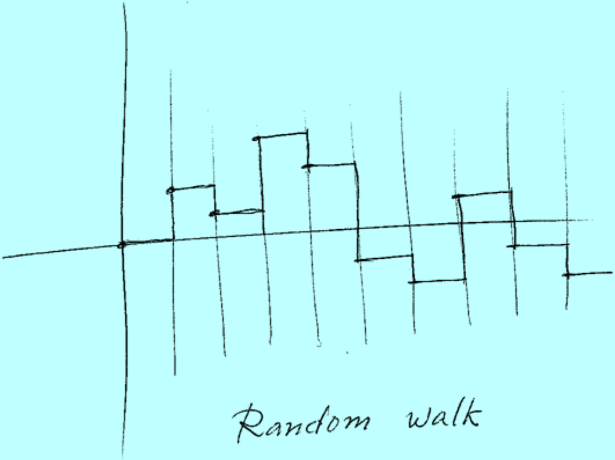
Distribution:

$$\nu(B) = P\{\eta \in B\} = P\{g(\xi) \in B\}$$

Summation:

$$X_n = X_0 + \eta_1 + \cdots + \eta_n$$

(Return to general case...)



Random walk

Transition kernel, Markov property

Conditional distribution:

$$\mu(x, B) = P[\, X_{n+1} \in B \,|\, X_n = x \,] = P\{f(x, \xi) \in B\}$$

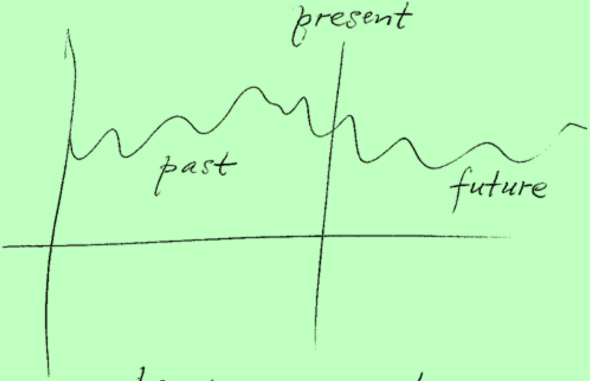
Conditioning on $(X_0, \ldots, X_n)$ gives same result !

Conditional independence:

$$(X_0, \ldots, X_{n-1}) \quad \underset{X_n}{\perp\!\!\!\perp} \quad (X_{n+1}, X_{n+2}, \ldots)$$

In words:

$$\text{(past)} \quad \underset{\text{(present)}}{\perp\!\!\!\perp} \quad \text{(future)}$$



Markov property

Dynamical system, continuous time

Simple recursion:

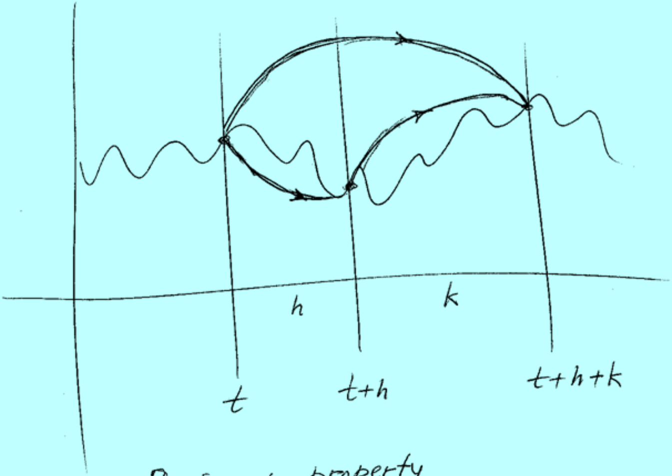
$$x_{t+h} = f_h(x_t), \quad t, h \geq 0$$

Iteration:

$$\begin{aligned} x_{t+h+k} &= f_k(x_{t+h}) \\ &= f_k \circ f_h(x_t), \quad t, h, k \geq 0 \end{aligned}$$

Semigroup property:

$$f_{s+t} = f_s \circ f_t, \quad s, t \geq 0$$



Semigroup property

Differential equation

Increments:

$$\frac{x_{t+h} - x_t}{h} = \frac{f_h(x_t) - f_0(x_t)}{h} = \frac{f_h - f_0}{h}(x_t)$$

Now let $h \longrightarrow 0$:

$$\frac{dx_t}{dt} = b(x_t), \quad t \geq 0$$

In differential form:

$$dx_t = b(x_t)\, dt$$

Dynamical system with randomness (Markov process)

Recursion:

$$X_{t+h} = f_h(X_t, \xi_t^{t+h}), \quad t, h \geq 0$$

Transition kernel:

$$\mu_h(x, B) = P[\, X_{t+h} \in B \,|\, X_t = x \,] = P\{f_h(x, \xi) \in B\}$$

Markov property:

$$\{X_s,\, s < t\} \quad \underset{X_t}{\perp\!\!\!\perp} \quad \{X_u,\, u > t\}$$

Semigroup property:

$$\mu_{s+t} = \mu_s \circ \mu_t, \quad s, t \geq 0$$

Generator:

$$\frac{\mu_h - \mu_0}{h} \to A$$

Space-homogeneous case (Lévy process)

Stationary, independent increments:

$$X_{t+h} - X_t = g_h(\xi_t^{t+h}), \quad t, h \geq 0$$

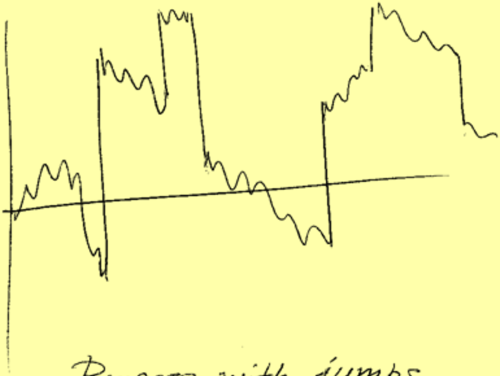
Increment distributions:

$$\nu_h(B) = P\{g_h(\xi) \in B\}, \quad h \geq 0$$

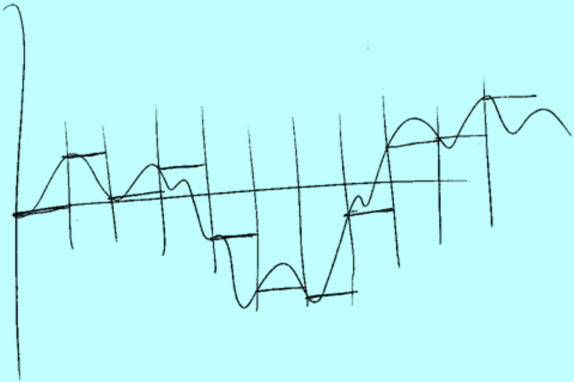
Semigroup property:

$$\nu_{s+t} = \nu_s * \nu_t, \quad s, t \geq 0$$

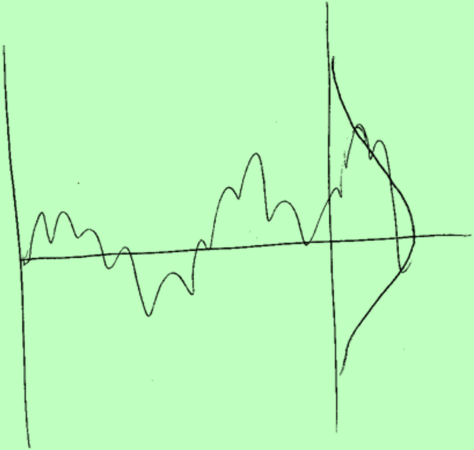
Paths may have jumps...



Process with jumps



Discrete skeleton



Normal approximation

Continuous paths (Brownian motion)

By central limit theorem and semigroup property:

$$X_t \sim N(b\,t,\, \sigma^2 t), \quad t \geq 0$$

for some constants:

b — drift

σ — diffusion rate

Now take $b = 0$ and $\sigma = 1$ (standardize):

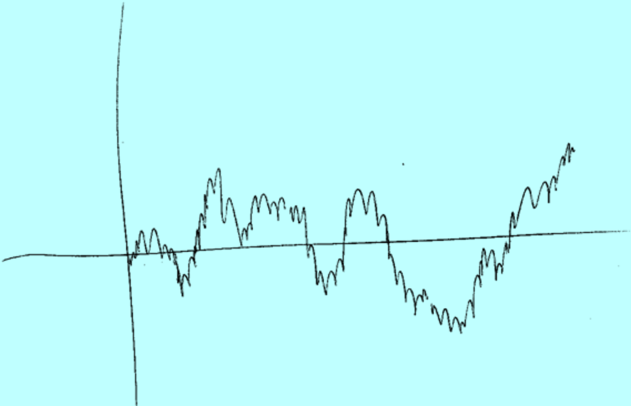
$$B_t \sim N(0, t), \quad t \geq 0$$

Then in general:

$$X_t = b\,t + \sigma B_t, \quad t \geq 0$$

In differential form:

$$dX_t = b\,dt + \sigma\,dB_t$$



Brownian motion

Stochastic differential equation, diffusion processes

Now let b and σ depend on location:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$$

In integrated form:

$$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s$$

The first integral is elementary. The second is not, since B is:

— nowhere differentiable

— has unbounded variation

— has extremely irregular paths

How to make sense of this ?

Itô stochastic integrals and SDE's

We can define the process:

$$(Y \cdot B)_t = \int_0^t Y_s \, dB_s, \quad t \geq 0$$

in a weak probabilistic sense, provided that:

— Y is non-anticipating (depends only on the past)

— $\int_0^t Y_s^2 \, ds < \infty, \quad t \geq 0$

Then we can prove existence and uniqueness of solutions to:

$$dX_t = b(X_t) \, dt + \sigma(X_t) \, dB_t$$

or more generally:

$$dX_t = b(t, X) \, dt + \sigma(t, X) \, dB_t$$

for suitable functions b and σ.

Itô's formula, stochastic calculus

Semimartingale decomposition:

$$X_t = X_0 + \int_0^t \sigma_s(X)\, dB_s + \int_0^t b_s(X)\, ds = X_0 + M_t + A_t$$

M_t — martingale part (centered process)

A_t — compensator (drift component)

Itô's formula (transformation of semimartingales):

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2} \int_0^t f''(X_s)\, d[X]_s$$

where $[X]_t$ denotes the quadratic variation of X_t.

In ordinary calculus, $[X]_t \equiv 0$, and the last term vanishes.

Reductions to Brownian motion

1. Diffusion $\rightarrow$ continuous martingale:

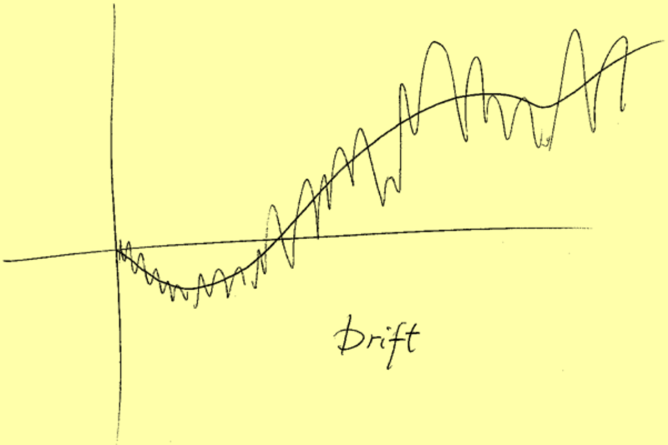
   $$X_t \text{ diffusion } \Rightarrow M_t = f(X_t) \text{ a continuous martingale}$$
   $$\text{for a suitable function } f$$

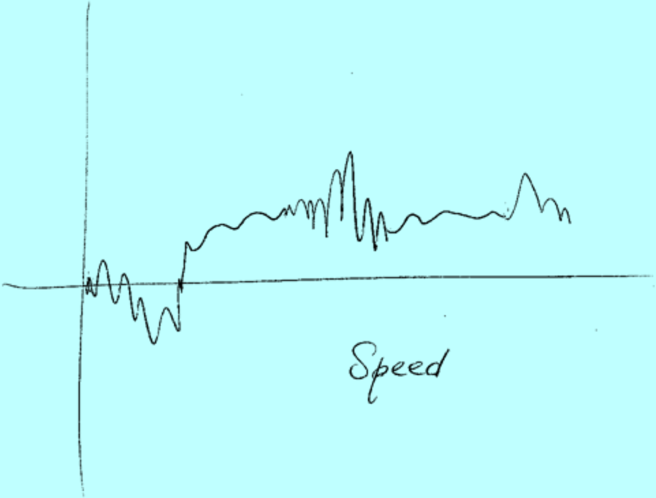
2. Continuous martingale $\rightarrow$ Brownian motion:

   $$M_t \text{ continuous martingale } \Rightarrow B_t = M \circ \tau_t \text{ a Brownan motion}$$
   $$\text{for a suitable process } \tau_t$$

Similarly, any point process can be time-changed into a Poisson process.

Thus, Brownian motion and Poisson processes emerge as the basic building blocks of stochastic processes.


Drift


Speed

Probability and potential theory

Diffusion (or heat) equation $\dot{u} = \frac{1}{2}\,\Delta u$, or:

$$2\,\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}$$

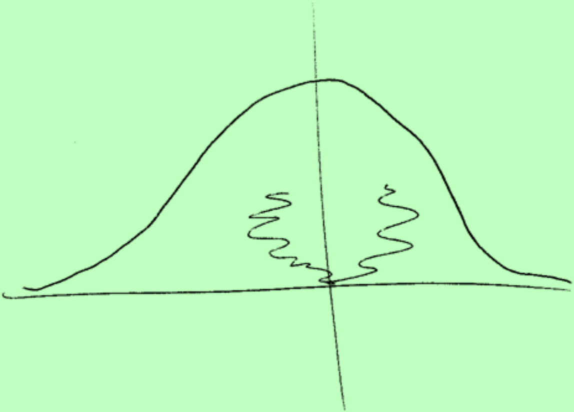
Fundamental solution:

$$u(x, t) = (2\pi t)^{-3/2} \exp(-|x|^2/2t)$$

This is also the probability density of $B_t \sim N(0, t)$.

- The PDE describes the *average* motion of a huge number of particles.
- Brownian motion describes the *individual* motion of each particle.

Connection: Brownian motion has generator $\frac{1}{2}\,\Delta$.



Heat kernel and Brownian motion

Some early history

Bachelier (1900–01) — random walk, Brownian motion

Markov (1906) — Markov property, Markov chains

Wiener (1923) — existence of Brownian motion

Bernstein (1927–37) — martingale property

Kolmogorov (1933–35) — conditioning, Markov processes

Lévy (1934–48) — Brownian paths, Lévy processes

Feller (1936–54) — Markov semigroups and generators

Doob (1940–53) — modern martingale theory

Itô (1942–51) — stochastic integration and SDE's

Dynkin (1955–56) — modern Markov process theory

Glossary of probability terms

stochastic — model involving randomness

stochastic process — randomly evolving function

probability theory — study of stochastic processes

Markov chain — recursion involving randomness

random walk — space-homogeneous random recursion

Markov process — stochastic dynamical system

diffusion — continuous Markov process

Brownian motion — space-homogeneous diffusion

Glossary of probability terms (continued)

Lévy process — space-homogeneous Markov process

Poisson process — independent-increment point process

Itô integral — stochastic integral w.r.t. Brownian motion

SDE — stochastic differential equation

martingale — process centered to have drift zero

potential theory — PDE-theory involving the Laplacian

semigroup — functions f_t satisfying $f_{s+t} = f_s \circ f_t$

A problem worthy of attack
Shows its worth by fighting back

— Pál Erdős

(Probability with tears)

Basic graduate course in probability

Discrete time (fall):

elements of measure theory

random variables and processes

conditioning, independence, 0–1 laws

strong limits, law of large numbers

weak limits, central limit theorem

martingales

Markov property and chains

Poisson and related processes

stationary processes and ergodic theory

Basic graduate course in probability (continued)

Continuous time (spring):

random walk, Brownian motion

Skorohod embedding, weak convergence

Markov processes and semigroups

Itô integrals and calculus

stochastic differential equations

continuous-time martingales

change of time, space, measure