What Are Graph Amalgamations?

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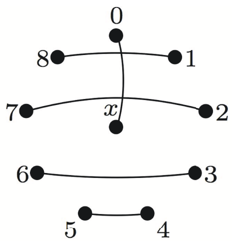
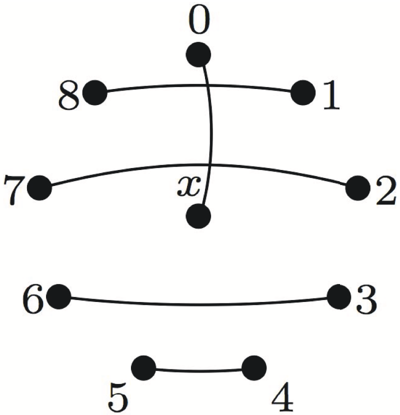
Factorizations

Suppose we have been entrusted to draw up a schedule for the "Big Ten" football teams. Each weekend they are to divide into 5 pairs and play. At the end of 9 weeks, we want every possible pair of teams to have played exactly once.



1-factorization of K_{10}



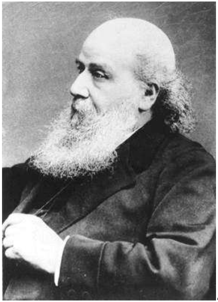
x•–•0	x•–•1	x•–•2	x•–•3	x•–•4	x•–•5	x•–•6	x•–•7	x•–•8
1•–•8	2•–•0	3•–•1	4•–•2	5•–•3	6•–•4	7•–•5	8•–•6	0•–•7
2•–•7	3•–•8	4•–•0	5•–•1	6•–•2	7•–•3	8•–•4	0•–•5	1•–•6
3•–•6	4•–•7	5•–•8	6•–•0	7•–•1	8•–•2	0•–•3	1•–•4	2•–•5
4•–•5	5•–•6	6•–•7	7•–•8	8•–•0	0•–•1	1•–•2	2•–•3	3•–•4

Sylvester's Problem

A set of $\frac{n}{h}$ h-subsets which partition the n-set $[n]$ is called a parallel class of h-subsets of $[n]$.

Question (Sylvester, 1847)

Can the set of all h-subsets be partitioned into parallel classes of h-subsets?



James Joseph Sylvester (1814–1897) (Source: Wikipedia)

Sylvester's Problem

Question (Sylvester, 1847)

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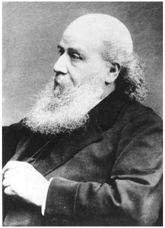
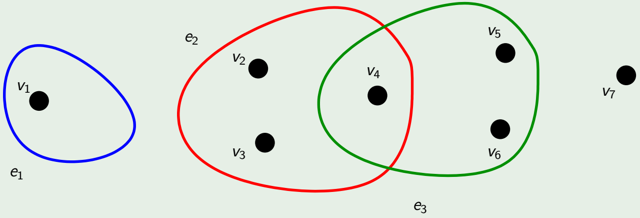


In 1877 Sylvester become the inaugural professor of mathematics at the new Johns Hopkins University in Baltimore, Maryland. His salary was $5,000 (quite generous for the time), which he demanded be paid in gold.

Hypergraphs

- A hypergraph $\mathcal{G} := (V, E)$, V is the vertex set, E is the edge (multi)set, every edge is a (multi)subset of V.

Example



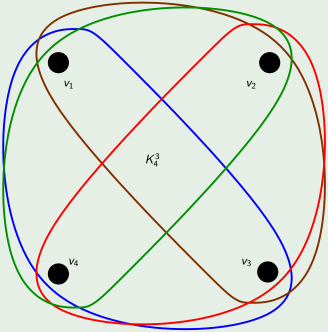
- r-factor: r-regular spanning,
- r-factorization: partition into disjoint r-factors,

Complete Uniform Hypergraphs

- $K_n^h = (V, \binom{V}{h})$: a complete h-uniform hypergraph on vertex set V with $|V| = n$.

Example

K_4^3



Baranyai's Theorem

Theorem

K_n^h is r-factorizable if and only if h divides rn and r divides $\binom{n-1}{h-1}$.


(Source: http://www.kfki.hu)

Baranyai was a Hungarian mathematician who was also a professional recorder player. He toured Hungary with the Barkfark Consort giving concerts and died in a car accident on a country road after one of them.

Baranyai's Theorem

Theorem

K_n^h is r-factorizable if and only if h divides rn and r divides $\binom{n-1}{h-1}$.

Zsolt Baranyai (1948–1978)

K_n^h is 1-factorizable if and only if h divides n. What if h doesn't divides n?

Baranyai-Katona Conjecture

Let m be the least common multiple of h and n, and let $a = m/h$. Define

$$\mathscr{K} = \{\{1, \ldots, h\}, \{h+1, \ldots, 2h\}, \ldots, \\ \{(a-1)h+1, (a-1)h+2, \ldots, ah\}\},$$

where the elements of the sets are considered mod n. The families obtained from $\mathscr{K}$ by permuting the elements of the underlying set $[n]$ are called *wreaths*.

- If h divides n, then a wreath is just a partition.
- If $\gcd(n, h) = 1$, then a wreath is a "Hamiltonian" cycle.

Conjecture

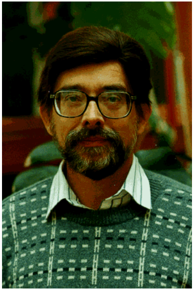
K_n^h can be decomposed into disjoint wreaths.

Connectivity

Conjecture

K_n^h can be decomposed into disjoint wreaths.

In connection with Baranyai-Katona conjecture, Katona suggested the problem of finding a connected factorization for K_n^h.



(Source: http://www.renyi.hu/ohkatona)

Connectivity

In connection with Baranyai-Katona conjecture, Katona suggested the problem of finding a connected factorization for K_n^h.

Theorem (B. 2011)

λK_n^h is $(r_1, \dots, r_k)$-factorizable if and only if h divides $r_i n$ for $1 \le i \le k$, and $\sum_{i=1}^{k} r_i = \lambda\binom{n-1}{h-1}$. Moreover, for $1 \le i \le k$, if $r_i \ge 2$, we can guarantee that the r_i-factor is connected.

While this generalizes Baranyai's result in various ways, we note that the major difference is connectivity. In particular if $\lambda = 1$, and $h = r_1 = \cdots = r_k = 2$, our result implies the classical result of Walecki that the edge set of K_n can be partitioned into Hamiltonian cycles if and only if n is odd.

Some Special Cases

Corollary

K_n^h is connected 2-factorizable if and only if $\binom{n-1}{h-1}$ is even, and h divides $2n$.

Corollary

K_n^h is connected $\frac{h}{\gcd(n,h)}$-factorizable.

Cameron's Problem

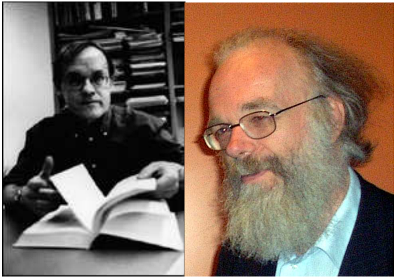
(1976) Under what conditions can partial 1-factorizations be extended to 1-factorizations?

(Source: http://www.math.uregina.ca/bailey)

When the partial edge-coloring is regular

Conjecture

(Baranyai-Brouwer-Schrijver (1976)) A 1-factorization of K_m^h can be extended to a 1-factorization of K_n^h iff h divides both m and n, and $n \geq 2m$.



(Source: http://techcn.com.cn, http://www.studeren.uva.nl)

(Baranyai-Brouwer- 1976) True for $h = 2$ and $h = 3$.

Haggkvist-Hellgren Theorem (1993)

Theorem

Every proper $\binom{m-1}{h-1}$-edge-coloring of K_m^h can be embedded in a proper $\binom{n-1}{h-1}$-edge-coloring of K_n^h iff h divides both m and n, and $n \geq 2m$.

(Source: http://www.umu.se)

Embedding r-factorizations

Theorem (B. & Rodger, to appear in J. Graph Theory)

Suppose that $n > 2m + \lfloor (1 + \sqrt{8m^2 - 16m - 7})/2 \rfloor$. A q-hyperedge-coloring of $\mathcal{F} = K_m^3$ can be embedded into an r-factorization of $\mathcal{G} = K_n^3$ if and only if

(i) *3 divides rn,*

(ii) *r divides $\binom{n-1}{2}$,*

(iii) *$q \leq \binom{n-1}{2}/r$, and*

(iv) *$d_j(v) \leq r$ for each $v \in V(\mathcal{F})$ and $1 \leq j \leq q$.*

Corollary

For $n \geq (2 + \sqrt{2})m$ the embedding problem is completely solved.

Embedding r-factorizations

Theorem (B. & Rodger, to appear in J. Graph Theory)

A k-hyperedge-coloring of $\mathcal{F} = K_m^3 \cup nK_m^2 \cup \binom{n}{2} K_m^1$ with $V = V(\mathcal{F})$ can be extended to an r-factorization of $\mathcal{G} = K_n^3$ if and only if

(i) 3 *divides* rn,

(ii) r *divides* $\binom{n-1}{2}$,

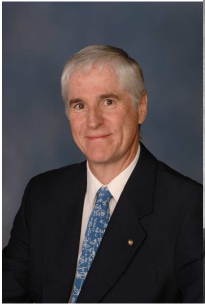
(iii) $k = \binom{n-1}{2}/r$,

(iv) $d_j(v) = r$ *for each* $v \in V$ *and* $1 \leq j \leq k$, *and*

(v) $|E^2(\mathcal{F}(j))| + 2|E^3(\mathcal{F}(j))| \geq r(m - n/3)$ *for* $1 \leq j \leq k$.

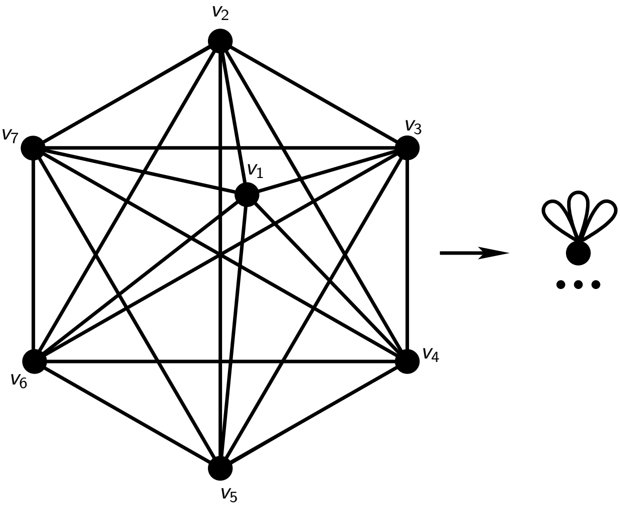
Amalgamations

Amalgamating a hypergraph $\mathscr{F}$ can be thought of as taking $\mathscr{F}$, partitioning its vertices, then for each element of the partition squashing the vertices to form a single vertex in the amalgamated graph $\mathscr{G}$.


(Source: http://www.personal.reading.ac.uk/ smshiltn/, http://ocm.auburn.edu, http://www-history.mcs.st-andrews.ac.uk)

Hamiltonian Decomposition of K_7: Amalgamation (Hilton 1984)



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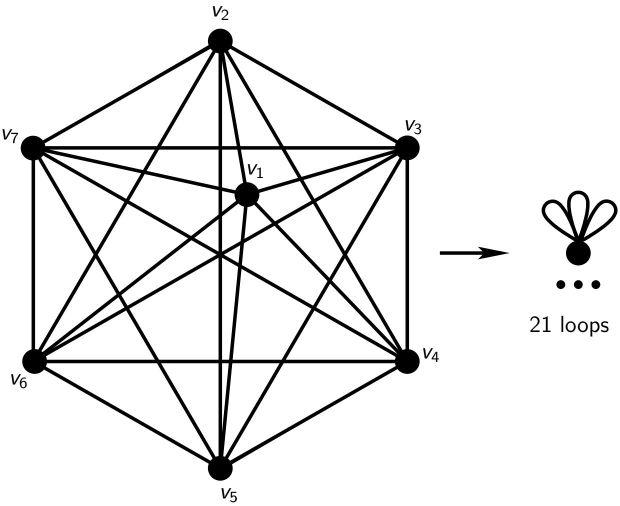
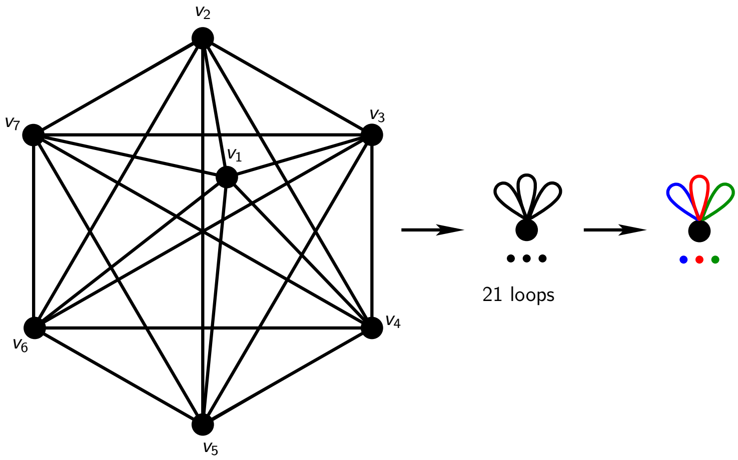
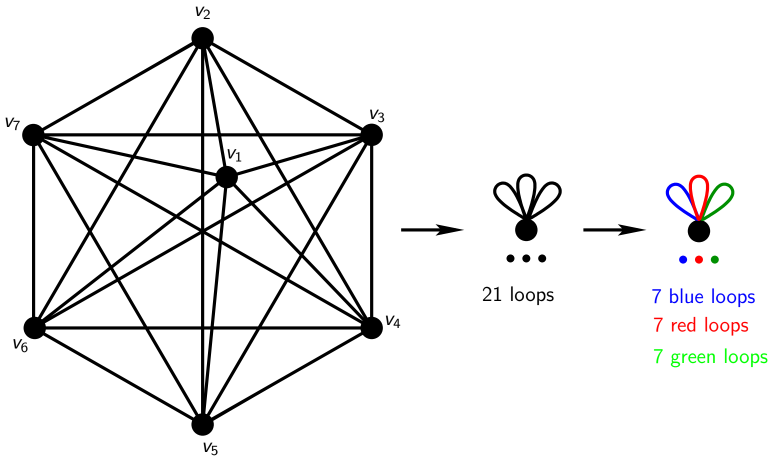


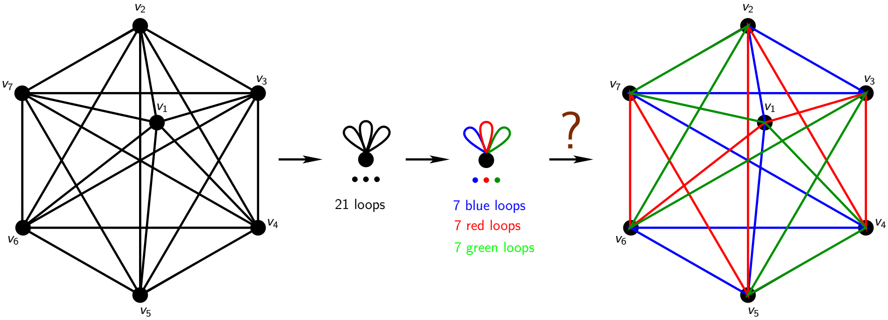
Hamiltonian Decomposition of K_7: Edge-coloring (Hilton 1984)



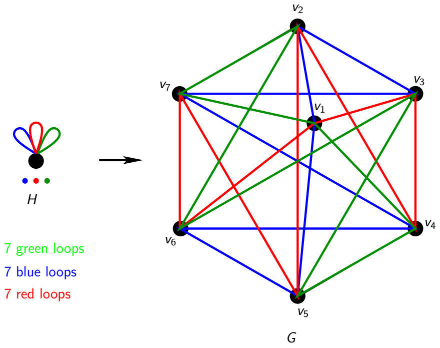
Hamiltonian Decomposition of K_7: Edge-coloring (Hilton 1984)



Hamiltonian Decomposition of K_7: Detachment (Hilton 1984)

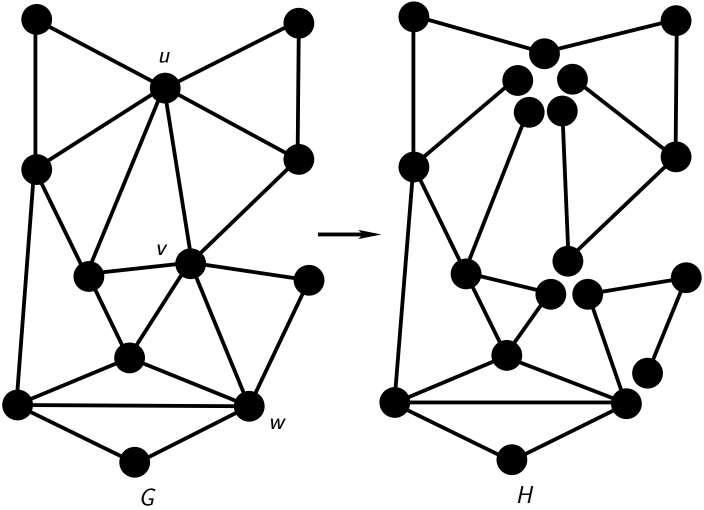


Hamiltonian Decomposition of K_7: Detachment (Hilton 1984)



- $d_{G(j)}(v_i) = 14/7 = 2$.
- each color class is connected.

Detachment



A Fair Detachment Theorem

Theorem (B., Rodger, to appear in J. Graph Theory)

H : k-edge-colored, $g : V(H) \to \mathbb{N}$. $\exists$ loopless g-detachment G of H such that for each distinct $w, z \in V(H)$, $\forall j \in \mathbb{Z}_k$:

(A1) $d_G(u) \approx d_H(w)/g(w)\ \forall u \in \psi^{-1}(w)$;

(A2) $d_{G(j)}(u) \approx d_{H(j)}(w)/g(w)\ \forall u \in \psi^{-1}(w)$;

(A3) $m_G(u, u') \approx \ell_H(w)/\binom{g(w)}{2}$ *when* $g(w) \geq 2\ \forall u, u' \in \psi^{-1}(w)$;

(A4) $m_{G(j)}(u, u') \approx \ell_{H(j)}(w)/\binom{g(w)}{2}$ *when* $g(w) \geq 2$, $\forall u, u' \in \psi^{-1}(w)$;

(A5) $m_G(u, v) \approx m_H(w, z)/(g(w)g(z))$ *for every pair of distinct vertices* $w, z \in V(H)$, *each* $u \in \psi^{-1}(w)$ *and each* $v \in \psi^{-1}(z)$;

(A6) $m_{G(j)}(u, v) \approx m_{H(j)}(w, z)/(g(w)g(z))\ \forall u \in \psi^{-1}(w)$, $\forall v \in \psi^{-1}(z)$;

(A7) *If for some* $j \in \mathbb{Z}_k$, $d_{H(j)}(w)/g(w)$ *is an even integer for each* $w \in V(H)$, *then* $\omega(G(j)) = \omega(H(j))$.

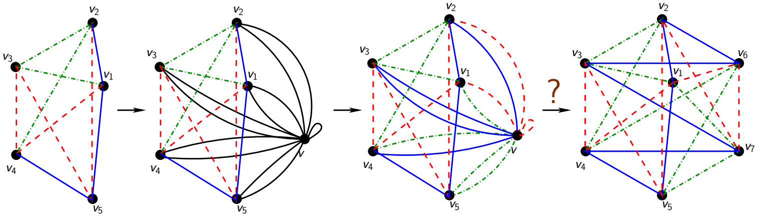
Applications

Theorem

(Walecki) λK_n *is Hamiltonian decomposable (with a 1-factor leave, respectively) if and only if* $\lambda(n-1)$ *is even (odd, respectively).*

Theorem

(Hilton) *A k-edge-colored K_m can be embedded into a Hamiltonian decomposition of K_{m+n} (with a 1-factor leave, respectively) if and only if $(m+n-1)$ is even (odd, respectively), $k = \lceil (m+n-1)/2 \rceil$, and each color class of K_m (except one color class, say k, respectively) is a collection of at most n disjoint paths, (color class k consists of paths of length at most 1, at most n of which are of length 0, respectively), where isolated vertices in each color class are to be counted as paths of length 0.*



Applications

Theorem

λK_n is $(r_1, \ldots, r_k)$-factorizable if and only if $r_i n$ is even for $1 \leq i \leq k$, and $\sum_{i=1}^{k} r_i = \lambda(n-1)$. Moreover, for $1 \leq i \leq k$ each r_i-factor can be guaranteed to be connected if r_i is even.

Theorem

A k-edge-coloring of K_m can be embedded into an $(r_1, \ldots, r_k)$-factorization of K_{m+n} if and only if $r_i(m+n)$ is even for $1 \leq i \leq k$, $\sum_{i=1}^{k} r_i = m+n-1$, $d_{K_m(i)}(v) \leq r_{\sigma(i)}$ for each $v \in V(K_m)$, $1 \leq i \leq k$, and some permutation $\sigma \in S_k$, and $|E(K_m(i))| \geq r_{\sigma(i)}(m-n)/2$.

Amalgamations of Hypergraphs, Hinge

Example

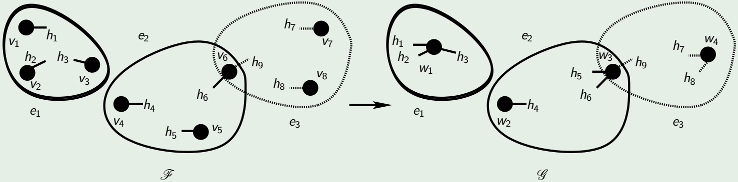
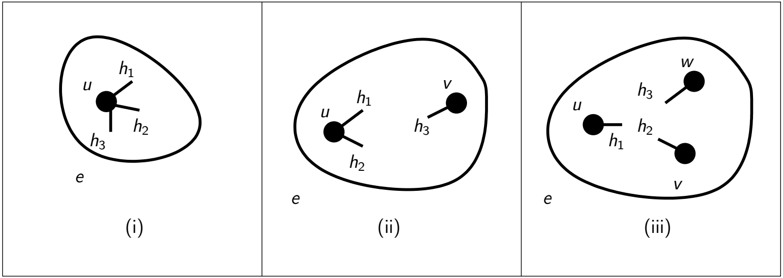


Figure: An amalgamation $\mathscr{G}$ of a hypergraph $\mathscr{F}$

$\mathscr{F}$ is a *detachment* of $\mathscr{G}$.

Notation

- $m(u^3)$,
- $m(u^2, v)$
- $m(u, v, w)$.



Theorem (B., to appear in J. Comb. Des.)

$\mathscr{F}$: *k-edge-colored, hypergraph,* $g : V(\mathscr{F}) \to \mathbb{N}$. $\exists$ *a 3-uniform g-detachment* $\mathscr{G}$ *of* $\mathscr{F}$ *with amalgamation function* $\Psi : V(\mathscr{G}) \to V(\mathscr{F})$:

(A1) $d_{\mathscr{G}}(u) \approx d_{\mathscr{F}}(x)/g(x)$ *for each* $x \in V(\mathscr{F})$ *and each* $u \in \Psi^{-1}(x)$;

(A2) $d_{\mathscr{G}(j)}(u) \approx d_{\mathscr{F}(j)}(x)/g(x)$ *for each* $x \in V(\mathscr{F})$, *each* $u \in \Psi^{-1}(x)$ *and each* $j \in \{1, \dots, k\}$;

(A3) $m_{\mathscr{G}}(u, v, w) \approx m_{\mathscr{F}}(x, y, z)/\tilde{g}(x, y, z)$ *for every* $x, y, z \in V(\mathscr{F})$ *with* $g(x) \geq 3$ *if* $x = y = z$, *and* $g(x) \geq 2$ *if* $|\{x, y, z\}| = 2$, *and every triple of distinct vertices* u, v, w *with* $u \in \Psi^{-1}(x)$, $v \in \Psi^{-1}(y)$ *and* $w \in \Psi^{-1}(z)$;

(A4) $m_{\mathscr{G}(j)}(u, v, w) \approx m_{\mathscr{F}(j)}(x, y, z)/\tilde{g}(x, y, z)$ *for every* $x, y, z \in V(\mathscr{F})$ *with* $g(x) \geq 3$ *if* $x = y = z$, *and* $g(x) \geq 2$ *if* $|\{x, y, z\}| = 2$, *every triple of distinct vertices* u, v, w *with* $u \in \Psi^{-1}(x)$, $v \in \Psi^{-1}(y)$ *and* $w \in \Psi^{-1}(z)$ *and each* $j \in \{1, \dots, k\}$.

Theorem (B., to appear in Comb. Prob. Comp.)

$\mathscr{F}$:k-edge-colored hypergraph, $g : V(\mathscr{F}) \to \mathbb{N}$. $\exists$ a g-detachment $\mathscr{G}$ of $\mathscr{F}$ with amalgamation function $\Psi : V(\mathscr{G}) \to V(\mathscr{F})$, st.

(A1) $d_{\mathscr{G}}(v) \approx d_{\mathscr{F}}(u)/g(u)$ for each $u \in V(\mathscr{F})$ and each $v \in \Psi^{-1}(u)$;

(A2) $d_{\mathscr{G}(j)}(v) \approx d_{\mathscr{F}(j)}(u)/g(u)$ for each $u \in V(\mathscr{F})$, each $v \in \Psi^{-1}(u)$ and $1 \leq j \leq k$;

(A3) $m_{\mathscr{G}}(U_1, \dots, U_r) \approx m_{\mathscr{F}}(u_1^{m_1}, \dots, u_r^{m_r})/\Pi_{i=1}^{r}\binom{g(u_i)}{m_i}$ for distinct $u_1, \dots, u_r \in V(\mathscr{F})$ and $U_i \subset \Psi^{-1}(u_i)$ with $|U_i| = m_i \leq g(u_i)$ for $1 \leq i \leq r$;

(A4) $m_{\mathscr{G}(j)}(U_1, \dots, U_r) \approx m_{\mathscr{F}(j)}(u_1^{m_1}, \dots, u_r^{m_r})/\Pi_{i=1}^{r}\binom{g(u_i)}{m_i}$ for distinct $u_1, \dots, u_r \in V(\mathscr{F})$ and $U_i \subset \Psi^{-1}(u_i)$ with $|U_i| = m_i \leq g(u_i)$ for $1 \leq i \leq r$ and $1 \leq j \leq k$.

2-edge-connected Fair Detachments

Theorem (B.)

Let $\mathscr{F}$ be a k-edge-colored (≤ 3)-hypergraph and let $g : V(\mathscr{F}) \to \mathbb{N}$ be a simple function. Then there exists a simple fair g-detachment whose color classes are all 2-edge-connected if and only if

$$\mathscr{F}(j) \text{ is 2-edge-connected for } 1 \leq j \leq k, \text{ and} \tag{1}$$

$$\frac{d_j(u)}{g(u)} \geq 2 \text{ for each } u \in V(\mathscr{F}), \text{ and } 1 \leq j \leq k. \tag{2}$$

THANKS