Name: ____________________

Applied Mathematics Preliminary Exam, Spring 2018

Instructions:

- This exam has eight problems; each problem is worth 10 points.
- Please choose any six and circle the problems that you choose. Do not circle more than six.
- Answer each problem on a new piece of paper.
- Write clearly and legibly.
- Answers will be graded on clarity and the correctness of the main steps of the reasoning.
- Though much effort has been made to eliminate typos and simple mistakes, if you notice one, ask the proctor.
- You may use the textbook *Applied Mathematics* by David Logan and a calculator.

1. For a pendulum that starts from rest, the period p depends on the length ℓ of the rod, on the gravity g, on the mass m of the ball, and on the initial angle θ_0 at which the pendulum is started.

   (a) Use dimensional analysis to determine the functional dependence of p on these four quantities.

   (b) For the largest pendulum ever built, the rod is 70 ft and the ball weighs 900 lbs. Assuming that $\theta_0 = \pi/6$ explain how to use a pendulum that fits on your desk to determine the period of this largest pendulum.

2. The following system represents the concentrations of two chemicals, x and y:

$$\frac{dx}{dt} = 1 - (r+1)x + x^2y$$
$$\frac{dy}{dt} = x(r - xy),$$

   where $r > 0$. Determine the local stability of the positive equilibrium and show that there is a Hopf bifurcation at $r = 2$.

3. Given that the boundary layer is at $x = 1$ find a uniform approximation to

$$\epsilon y'' - 3y' - y^4 = 0, \text{ for } 0 < x < 1,$$

   where the boundary conditions are $y(0) = 1$ and $y(1) = 1$.

4. Find the extremals for the functional

$$J(y) = \int_0^\pi (y')^2 + x^2 dx \text{ subject to the constraint } \int_0^\pi y^2 dx = 1$$

   where $y(0) = 0$ and $y(\pi)$ is free.

5. Solve the integral equation

$$u(x) = f(x) + \int_0^1 (1 + e^{x+y})u(y)dy.$$

6. Use the Laplace transform to solve

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad 0 < x < \ell, \quad t > 0, \\ u(0,t) &= u(\ell,t) = 0, \quad t > 0, \\ u(x,0) &= \sin(\pi x/\ell), \quad u_t(x,0) = -\sin(\pi x/\ell). \end{aligned}$$

7. An activator-inhibitor reaction-diffusion system in dimensionless form is given by

$$\begin{aligned} u_t &= u_{xx} + \frac{u^2}{w} - \frac{1}{2}u, \\ w_t &= d w_{xx} + u^2 - w, \end{aligned}$$

where $0 < x < \pi$, $t > 0$, and d is a positive constant. Examine the stability of the constant steady state of this system under no-flux boundary conditions.

8. The females of a certain population can be classified as either juveniles or adults. Suppose that the juveniles do not reproduce, the juvenile stage lasts only one year, and adult females may live longer than one year. You may assume that the fecundity and the survivorship of adult females is independent of their age.

   (a) Set up a Leslie model to describe the dynamics of the females in this population.

   (b) Identify the growth rate and determine the long time proportion of the female population that are juveniles.