PRELIMINARY EXAMINATION

Real Analysis　　　　　　　　　　Summer 08
Time: 3 hours　　　　　　　　　　August 02, 2008

Note 1: Please **solve 3 problems from part 1**, **3 problems from part 2** and **2 problem from part 3**

Note 2: You may solve as many problems you possibly can, if you have the time

Part 1

1. State the Radon-Nikodyn theorem and give an application. Elaborate your application.

2. Let $L_1[0,1]$, the Lebesgue space of integrable functions on the interval [0,1] with the Lebesgue measure, and let φ be a bounded linear functional on $L_1[0,1]$. Define the function by g(x) = $\varphi(\chi_{[0,x]})$. Show that g is absolutely continuous on the interval [0,1].

3. a) Let X de a normed space and $T : R^n \to X$ be a linear transformation. Show that there is a positive constant M, so that $\|Tx\|_X \le M\|x\|_{R^n}$

   b) Let $\| \ \|_1$ and $\| \ \|_2$ be any two norms in R^n. Show that there are positive constants M, N So that $M\|x\|_1 \le \|x\|_2 \le N\|x\|_1$.

   c) φ is a bounded linear functional on R^n if and only if there is a unique $y \in R^n$, so that $\varphi(x) = \sum_{i=1}^{n} x_i y_i$, and $\|\varphi\| = \|y\|_{R^n}$.

4. let (X, A, μ) be a measure space and $L_p = L_p(X, \mathrm{A}, \mu)$. Given $p, r \in [1,\infty)$ with p≥r and a Measurable function g. Show that $T : L_p \to L_r$ defined by T(f) = g.f is bounded if and only if $g \in L_s$ where $\frac{1}{s} = \frac{1}{r} - \frac{1}{p}$.

5. A sequence (f_n) of real valued measurable functions is said do converges in measure to a function f is for every $\varepsilon > 0$ $\lim_{n\to\infty} \mu\{x \varepsilon X : |f_n(x) - f(x)| \ge \varepsilon\} = 0$.

   If $\mu(X) < \infty$, show that a sequence (f_n) converges in measure to a function f, if and only if $r(f_n - f) \to 0$, as n tends to ∞, where $r(f) = \int_X \frac{|f(x)|}{1+|f(x)|} d\mu(x)$.

Part 2

1. (a) State Heine-Borel Theorem and use it to prove that if f is a real-valued function defined and continuous on a closed and bounded set F of real numbers then f is uniformly continuous on F.

   (b) Let $< f_n >$ be a sequence of continuous functions defined on a set E. Prove that if $< f_n >$ converges uniformly to f on E, then f is continuous on E.

2. Let $E \subset [0,1]$ be a measurable set and let $E \dot{+} y = \{z : z = x \dot{+} y \text{ for some } x \in E\}$ be the translate modulo 1 of E. Prove that for each $y \in [0,1]$, the set $E \dot{+} y$ is measurable and $m(E \dot{+} y) = mE$. Use this result to construct a nonmeasurable set.

3. (a) If $\{A_n\}$ is a countable collection of sets of real numbers, then prove that

$$m^*(\bigcup A_n) \leq \sum m^* A_n.$$

   Use the above result to prove that the set $[0,1]$ is uncountable.

   (b) Let f be a measurable function and $f = g$ almost everywhere. Prove that the function g is measurable.

4. (a) State and prove Fatou's Lemma and use this to prove Lebesgue Monotone Convergence Theorem, that is, if $< f_n >$ is an increasing sequence of nonnegative measurable functions $f(x) = \lim_{n\to\infty} f_n(x)$ a.e., then

$$\int f = \lim_{n\to\infty} \int f_n.$$

   (b) Let f be a nonnegative function which is integrable over a set E. Then prove that for given $\epsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have

$$\int_A f < \epsilon.$$

Part 3

1. Consider a series $\sum_{k=1}^{\infty} a_k(x)$, where the functions $a_k(x) \ge 0$ are measurable for every k≥1. Prove that $\int_R \sum_{k=1}^{\infty} a_k(x)dx = \sum_{k=1}^{\infty} \int_R a_k(x)dx$.

   If $\sum_{k=1}^{\infty} \int_R a_k(x)dx$ is finite, prove that he series $\sum_{k=1}^{\infty} a_k(x)$ converges for almost everywhere.

2. Suppose f is a non-negative measurable function, and (f_n) a sequence of non-negative measurable functions with $f_n(x) \le f(x)$ and $f_n(x) \to f(x)$ almost everywhere as $n \to \infty$. Prove that $\lim_{n\to\infty} \int_R f_n(x)dx = \int_R f(x)dx$.

3. Suppose f is integrable on R^n. Prove that for every $\varepsilon > 0$,
   a) There exists a set of finite measure B (a ball, for example) such that $\int_{B^c} |f(x)|dx < \varepsilon$.

   b) There is a $\delta > 0$ such that $\int_E |f(x)|dx < \varepsilon$ whenever $|E| < \delta$, where $|E|$ denotes the measure of the set E.