Graph Theory Prelim – 2008

1. In each of the following, prove or disprove the assertion.
   a. Every connected simple graph G has a spanning tree with the same maximum degree as G.
   b. Every connected simple graph G has a spanning tree with the same minimum degree as G.
   c. Every connected simple graph G has a spanning tree with the same domination number as G.
   d. Every connected simple graph G has a spanning tree with the same vertex independence number as G.

2. Recall that a vertex-covering of G is a set of vertices S such that each edge in G is incident with a vertex in S. For a simple graph G, let $\acute{\alpha}(G)$ denote the vertex independence number, let $\beta(G)$ denote the vertex-covering number, let $\acute{\alpha}'(G)$ denote the edge-independence number and $\beta'(G)$ denote the edge-covering number. Let the number of vertices in G be n.

   a. Show that $\acute{\alpha}(G) + \beta(G) = n$. (Hint: Consider complements of independent sets and of coverings.)

   b. Show that if G has no isolated vertices then $\acute{\alpha}'(G) + \beta'(G) = n$.

   c. If G is a tree with exactly 14 edges, and $\acute{\alpha}'(G) = 4$, then find $\acute{\alpha}(G)$, $\beta(G)$ and $\beta'(G)$, giving reasons for your answers. (Hint: For bipartite graphs there is also an equality that involves 2 more of these 4 parameters that you probably know.)

3. Throughout this question, all graphs are simple.

   a. For positive integers s, t, u with $s \leq t \leq u$, describe a graph with $\kappa = s$, $\kappa' = t$, and $\delta = u$.

   b. Show that $\kappa' \leq \delta$.

   c. Let the number of vertices in G be n. Show that if $\delta(G) \geq n/2$ then $\kappa' = \delta$. (Hint: count the sum of the degrees of the vertices in C in two ways, where C is a smallest component after a minimum edge-cut is removed from G.)

4. Let B be a bipartite graph with $\delta \geq 1$.

   a. Show that there exists an edge-coloring of B with δ colors such that for each vertex v and for each color c there exists an edge incident with v colored c.

   b. Show that if B is k-regular for some positive integer k, and if there exist positive integers $k_1, \ldots, k_x$ which add to k, then there exists an edge-coloring of B such that the i^{th} color class is a k_i-factor for $1 \leq i \leq x$.

   c. Show that the property in Question 4b is not necessarily true if B is not bipartite.

5. Show that:

   a. If G is Hamiltonian then for all subsets S of V(G) the number of components in G-S is at most |S|.

   b. Suppose G is a connected simple graph containing a path P of length $s \leq n-2$, where n is the number of vertices in G. Show that if the sum of the degrees of the first and last vertices in P is at least n, then G contains a path of length $s+1$.