Dr. Paul G. Schmidt August 14, 2010

Preliminary Doctoral Examination in Analysis

Name: ______________________________

Work five of the following six problems. Each is worth 20 points (10 for each part).

You may work the remaining problem for a bonus. If you do so, please indicate clearly which is the bonus problem (else the sixth problem you work will be considered your bonus problem).

Please note that no books, notes, or electronic communication devices are allowed during the exam.

1. Let X be a nonempty set, $\mathcal{M}$ a nonempty collection of subsets of X, and μ a mapping from $\mathcal{M}$ into $[0,\infty]$.

(a) Define what it means to say that $\mathcal{M}$ is a σ-algebra. Assuming that $\mathcal{M}$ is a σ-algebra, define what it means to say that μ is a measure. Further, assuming that $\mathcal{M}$ is a σ-algebra and μ is a measure, define what it means to say that μ is σ-finite.

(b) Assuming that X is uncountable, let $\mathcal{M}$ denote the collection of all subsets of X that are countable or have a countable complement. For $A \in \mathcal{M}$, define $\mu(A) := 0$ if A is countable, $\mu(A) := \infty$ if $X \setminus A$ is countable. Show that $\mathcal{M}$ is a σ-algebra and μ is a measure.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) State the Monotone Convergence Theorem, Fatou's Lemma, and the Dominated Convergence Theorem (complete with all assumptions and assertions).

(b) Use the Monotone Convergence Theorem to prove Fatou's Lemma. Alternatively, use Fatou's Lemma to prove the Dominated Convergence Theorem.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $p \in [1,\infty)$.

(a) Give the definition of the space $L^p = L^p(X, \mathcal{M}, \mu, \mathbb{K})$ and its norm, $\|\cdot\|_p$.

(b) Prove that "L^p-dominated convergence a.e." implies "convergence in L^p." That is, if a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable, $\mathbb{K}$-valued functions on X converges almost everywhere to a function f, and if there exists a function $g \in L^p$ such that $|f_n| \le g$ almost everywhere for every $n \in \mathbb{N}$, then $f_n \in L^p$ for every $n \in \mathbb{N}$, $f \in L^p$, and $\|f_n - f\|_p \to 0$ as $n \to \infty$.

4. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let f be a $(\mu \times \nu)$-integrable function on $X \times Y$ (extended real-valued or complex-valued).

(a) State Fubini's Theorem (complete with all assertions, under the above assumptions).

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) := e^{-xy}\sin(x)$. Use Tonelli's Theorem to show that f is Lebesgue-integrable on $(0,\infty) \times (1,\infty)$. Then apply Fubini's Theorem to compute the integral of f over $(0,\infty) \times (1,\infty)$ in two ways and infer that $\int_0^\infty e^{-x}\,\frac{\sin(x)}{x}\,dx = \frac{\pi}{4}$. (Justify all your computations!)

Hints for Part (b): $|\sin(x)| \le |x|$ for all $x \in \mathbb{R}$; $\int e^{-xy}\sin(x)\,dx = -\frac{e^{-xy}}{1+y^2}\big(y\sin(x) + \cos(x)\big) + \text{const}$ for all $x, y \in \mathbb{R}$

5. Suppose G and H are open subsets of $\mathbb{R}^n$, for some $n \in \mathbb{N}$, and $T : G \to H$ is a diffeomorphism.

(a) Assuming that f is a Lebesgue-measurable function on H, nonnegative or Lebesgue-integrable, express the integral $\int_H f(y)\, dy$ as an integral over G ("Change-of-Variables Formula").

(b) Let λ_n denote n-dimensional Lebesgue measure; for $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B_n(x, r)$ denote the open n-dimensional ball of radius r centered at x. Show that $\lambda_n(B_n(x, r)) = r^n \lambda_n(B_n(0, 1))$ for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$.

6. Let $(X, \mathcal{M})$ be a measurable space and let μ and ν be measures on $\mathcal{M}$.

(a) Define what it means for ν to be absolutely continuous with respect to μ. Further, define what we mean by a density of ν with respect to μ. Then state the Radon-Nikodym Theorem (complete with all assumptions and assertions).

(b) Given $n \in \mathbb{N}$, let $\mathcal{L}_n$ and λ_n denote the Lebesgue σ-algebra and Lebesgue measure in $\mathbb{R}^n$; for $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B_n(x, r)$ denote the open ball of radius r centered at x. Suppose that ν is a measure on $\mathcal{L}_n$ with $\nu((B_n(x, r)) < \infty$ for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$ and that ν has a density f with respect to λ_n. Show that, for λ_n-almost every $x \in \mathbb{R}^n$, $f(x) = \lim_{r \to 0} \frac{\nu(B_n(x,r))}{\lambda_n(B_n(x,r))}$.