Invariants

$G = \text{group acting on a space } X.$

$G$ acts on the complex-valued functions on $X$ by:

$$g^*(f)(x) = f(g^{-1}(x)).$$

This operation preserves the algebra operations on functions - addition and multiplication.

A function $f$ on $X$ is \textit{invariant} if it is unchanged by the action.

Because of the algebra-preserving properties of $g^*$, the sum of two invariant functions is again invariant, and the product of two invariant functions is again invariant. Of course, constant functions are invariant, so also, a constant multiple of an invariant function is invariant. We say that the invariant functions form an \textit{algebra}.

The basic problem of invariant theory is to describe this algebra.

\textbf{Geometric motivation:} Knowing the invariants gives information on the $G$-orbits:

If two points are in the same orbit, they will give the same values for all invariants.

Thus the invariants help to separate the orbits.
Examples:

Rotating Conic Sections.

The group of rotations of $\mathbb{R}^2$ is the set of matrices

$$\rho_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

If $F(x, y) = Ax^2 + 2Bxy + Cy^2$ is a homogeneous quadratic polynomial, then the equation $F(x, y) = 1$ defines a conic section. By an appropriate rotation, $F$ can be put in the standard form

$$\rho_\theta^* F = \frac{x^2}{a^2} \pm \frac{y^2}{b^2}.$$ 

However, it can be shown that the quantity $AC - B^2$ is invariant under rotations. So we have

$$AC - B^2 = \pm \frac{1}{ab}.$$ 

This shows:

i) The curve $F(x, y) = 1$ is a hyperbola if $AC - B^2 < 0$, and is an ellipse if $AC - B^2 > 0$ (and $A > 0$).

ii) When the curve is an ellipse, the area of the interior of the ellipse is

$$Area = \pi ab = \pi \frac{1}{\sqrt{AC - B^2}}.$$ 

Moral: Invariants carry interesting geometric information.
Coefficients of the Characteristic Polynomial.

The general linear group $GL_n$ of all invertible $n \times n$ matrices acts on the space $M_n$ of matrices by conjugation:

$$\gamma_g(T) = gTg^{-1}. \quad g \in GL_n; T \in M_n$$

The coefficients of the characteristic polynomial

$$\det(T - \lambda I) = \sum_{\ell=1}^{n} (-1)^{\ell} c_{\ell}(T) \lambda^{\ell}$$

are invariant under the conjugation action $\gamma$.

Key result of linear algebra: the characteristic polynomial of $T$ determines the eigenvalues of $T$.

**Theorem:** The coefficients $c_{\ell}$ of the characteristic polynomial generate the algebra of polynomial invariants for $GL_n$ acting on $M_n$ by conjugation.
Four Stages of Classical Invariant Theory

1. Weyl's Fundamental Theorems.

2. Transfer to the Weyl Algebra; Dual Pairs; Generalized Spherical Harmonics

3. Seesaw pairs; Reciprocity Laws and Reciprocity Algebras

4. Kernel of the Harmonic Decomposition
The Classical Groups

1. The general linear group $GL_n$.

This is the group of invertible $n \times n$ matrices, or equivalently, the group of all invertible linear transformations of an $n$-dimensional vector space $V$.

It acts on $V \cong \mathbb{C}^n$, consisting of column vectors of length $n$, by the usual row-by-column multiplication. It also acts on the space $(\mathbb{C}^n)^*$ of row vectors $\lambda = (y_1, y_2, y_3, \ldots, y_n)$ again essentially by row-by-column multiplication, but with a slight modification:

$$g^*(\lambda) = \lambda g^{-1}$$

There is a natural pairing $\alpha : V \times V^* \rightarrow \mathbb{C}$, given by

$$\alpha(\vec{v}, \vec{\lambda}) = \vec{\lambda}(\vec{v}) = \vec{\lambda} \cdot \vec{v},$$

where $\cdot$ indicates row-by-column matrix multiplication. This pairing is invariant for the action of $GL(V)$ on the two spaces:

$$g^*(\vec{\lambda})(g(\vec{v})) = \vec{\lambda}(\vec{v}),$$

for all $g \in GL_n$, $\vec{v} \in \mathbb{C}^n$ and $\vec{\lambda} \in (\mathbb{C}^n)^*$. 
Let $B(\vec{x}, \vec{y})$ be an symmetric inner product on a vector space $V$. For example, if $V = \mathbb{R}^n$, and

\[
\vec{x} = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\vdots \\
x_n
\end{bmatrix}, \quad \text{and} \quad \vec{y} = \begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_n
\end{bmatrix},
\]

then we could take

\[
B_\circ(\vec{x}, \vec{y}) = \langle \vec{x}, \vec{y} \rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n = \sum_{j=1}^{n} x_jy_j
\]

that is, the standard Euclidean inner product. We could extend this to $\mathbb{C}^n$ by the same formula. We assume that $B$ is non-degenerate, in the sense that, for any $\vec{x}$, there is some $\vec{y}$ such that $B(\vec{x}, \vec{y}) \neq 0$.

The orthogonal group $O_B$ is the group of all linear transformations $g$ that preserve $B$, in the sense that

\[
B(g(\vec{x}), g(\vec{y})) = B(\vec{x}, \vec{y})
\]

for all pairs of vectors $\vec{x}$ and $\vec{y}$ in $V$.

For the standard inner product,

\[
O_{B_\circ} = \{ g \in GL_n : gg^T = I_n \},
\]

where $g^T$ denotes the usual matrix transpose of $g$. 

\[
\]
3. The symplectic group $Sp_{2n}$.

We can consider, instead of a symmetric bilinear form, a skew-symmetric one. For this to be non-degenerate requires the dimension to be even. Thus on $\mathbb{R}^{2n}$, we can consider the non-degenerate, skew-symmetric (= symplectic) bilinear form

$$\langle \vec{x}, \vec{y} \rangle = \sum_{j=1}^{n} x_j y_{n+j} - x_{n+j} y_j = B_0(\vec{x}, J\vec{y}),$$

where

$$J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix},$$

where $0_n$ and $I_n$ are respectively the $n \times n$ zero matrix, and the $n \times n$ identity matrix. Then the symplectic group $Sp_{2n}$ is the set of isometries $g$, of the form $\langle \cdot, \cdot \rangle$, in the sense that

$$\langle g(\vec{x}), g(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle,$$

again for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^{2n}$. Here, it does not matter if the field of scalars is $\mathbb{R}$ or $\mathbb{C}$, or whatever, there is only one symplectic group in a given dimension.
The classical actions

The action of $G = GL_n$, or $O_n$, or $Sp_{2n}$ on the vector space $V$ used to define it is called the standard action of $G$. We can let $G$ also act on $V^m$, the sum of $m$ copies of $V$, by the coordinate-wise action. If $V \cong \mathbb{C}^n$, then we can think of $V^m$ as the $n \times m$ matrices, with each of $m$ columns constituting a copy of $V$. Then the action of $G$ on $V^m$ can be accomplished by ordinary matrix multiplication (on the left).

We can also let $G$ act on the dual space $V^*$, as described above for the general linear group.

By a classical action of $G$, we mean the action of $G$ on a direct sum $V^p \oplus (V^*)^q$. For the orthogonal and symplectic groups, it is easy to show that the actions of $G$ on $V$ and on $V^*$ are equivalent, so it is not necessary to use any copies of $V^*$.

Note that:

i) The direct sum of two classical actions is again a classical action.

ii) The dual of a classical action is again a classical action.
Stage I

Weyl’s First Fundamental Theorem

Invariants for Classical Actions.

Weyl’s First Fundamental Theorem says:

the obvious invariants are all the invariants.

More precisely:

Let $G = GL(V)$ act on $Y = Y_{p,q} = V^p \oplus (V^*)^q$. Write $\vec{z} = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p, \vec{\lambda}_1, \vec{\lambda}_2, \ldots, \vec{\lambda}_q)$. Then

$$r_{ab}(\vec{z}) = \vec{\lambda}_a(\vec{x}_b)$$

will be invariant for the action of $GL(V)$ on $Y$. In the usual coordinates on $Y$, this will be a quadratic function. Because of the invariance of the pairing between $V$ and $V^*$, the $r_{ab}$ will be invariant under the action of $GL_n$ on $V$.

Similarly, if $G = O_B$, then for $\vec{z} = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p)$, define

$$r_{ab} = B(\vec{x}_a, \vec{x}_b).$$

For obvious reasons, this will be an invariant for the action of $O_B$ on $V^p$. As in the case of $GL_n$, the $r_{ab}$ will be

In a parallel fashion, we can also define functions $r_{ab}$ for the action of $Sp_{2n}$ on $V^p$, where here $V = \mathbb{C}^{2n}$, using the symplectic pairing, and these $r_{ab}$ will again be quadratic functions in the coordinates on $V^p$, and will be invariant for the action of $Sp_{2n}$.

First Fundamental Theorem of Invariants:

Let $Y$ be a classical action of the classical group $G$. Let $r_{ab}$ denote the quadratic invariants defined above. Then the full algebra $P(Y)^G$, of polynomials on $Y$ that are invariant under $G$, is generated by the $r_{ab}$. 
The Yoga of Lie Groups

A Lie group $G$ is a group that is also a smooth manifold. (and right and left translations are smooth maps)
The Lie algebra $g$ of $G$ is the tangent space to $G$ at 1, the identity element. (also describable as left invariant vector fields)
The Lie algebra has a canonically defined skew-symmetric bilinear product (aka bracket operation):

$$[\cdot,\cdot] : g \times g \to g$$

There is canonical map (the exponential map):

$$\exp : g \to G$$

For each $x$ in $g$, the group elements $\exp(tx)$, for real numbers $t$, form a one-parameter group:

$$\exp(sx)\exp(tx) = \exp((s + t)x).$$

Given a representation of $G$ (a homomorphism $\rho : G \to GL(V)$ on a vector space $V$, we get a (derived) representation $d\rho$ of $g$ by

$$d\rho(x) = \frac{d}{dt}\rho(\exp(tx))|_{t=0}.$$ 

This preserves bracket:

$$[d\rho(x),d\rho(y)] = d\rho(x)d\rho(y) - d\rho(y)d\rho(x) = d\rho([x,y]).$$

Moreover, $d\rho$ determines $\rho$ (on the identity component of $G$) by the recipe

$$\rho(\exp(tx)) = \exp(t d\rho(x)),$$

where the second exp is the standard exponential map on matrices:

$$\exp A = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \ldots + \frac{A^k}{k!} + \ldots$$

Thus, we can move back and forth between:

representations of a (connected) Lie group

and

representations of its Lie algebra.
Background on the Weyl Algebra

The Weyl Algebra, IA: the CCR

Key insight of Heisenberg:

At the microscopic level, the operations of measuring momentum and position are not interchangeable - they do not commute with one another.

Instead, Heisenberg proposed that, if $p_j$ are the momenta of the particles in a system, and $q_j$ are the positions, that (in a sanitized, unitfree form for mathematicians)

$$[p_j, q_k] = p_j q_k - q_k p_j = \delta_{jk} I,$$

where $\delta_{jk}$ is Kronecker’s delta, and $I$ is the identity operator.

These are the Heisenberg Canonical Commutation Relations (CCR).
The Weyl Algebra, IB: representing the CCR

The coordinate functions $x_j$ generate the algebra $P(C^n)$ of polynomial functions on the vector space $C^n$. They can be regarded as operators on $P(C^n)$ by multiplications.

In a similar fashion, the operators

$$D_k = \frac{\partial}{\partial x_k}$$

of partial differentiation in the coordinate directions, generate the algebra $S(C^n)$ of all translation-invariant differential operators on $C^n$. These also act on $P(C^n)$.

The operators $x_j$ and $D_k$ together generate the algebra of all polynomial coefficient differential operators on $P(C^n)$. We call this algebra $W(C^n) = W_n$, the Weyl algebra.
The CCR can be formulated more geometrically. Let $X_n = X =$ linear span of the $x_j$, and $Y_n = Y =$ linear span of the $D_j$, and $W_n = W = X \oplus Y$.

The commutator defines a bilinear product of $W_n$ to itself, and this product is skew-symmetric. The relations (CCR) say that the commutator of any two element of $W_n$ will be a scalar operator - a multiple of the identity. Thus we can write

$$[w, w'] = \langle\langle w, w' \rangle\rangle \iota,$$

where $\iota$ denotes the identity operator on $P(\mathbb{C}^n)$. The function $\langle\langle w, w' \rangle\rangle$ is easily checked to be a symplectic (= bilinear, skew-symmetric, non-degenerate) form on $W_n$.

The CCR exactly capture the structure of $W$, in the following sense:

**Theorem:** The Weyl algebra $W$ of polynomial coefficient differential operators is the universal associative algebra generated by $W$, subject to the CCR.
The Weyl Algebra, II: Conjugation by $GL_n$

$GL_n$ acts on $P(C^n)$ according to the general formula on page 3. The operators of $GL_n$ do not belong to $W_n$, but the associated infinitesimal action of the Lie algebra $gl_n \simeq M_n$, the $n \times n$ matrices, does belong to $W_n$. Call the action $\alpha$. A calculation shows that if $T = \{t_{kj}\}$ is an $n \times n$ matrix, and $p$ is a polynomial, then

$$\alpha(T)(p) = -\sum_{j,k=1}^{n} t_{kj} x_j D_k(p)$$

Although $GL_n$ does not belong to $W_n$, it does act on $W_n$ by conjugation: If $L$ is a polynomial coefficient differential operator, and $g$ is in $GL_n$, then

$$Adg(L) = g^* L(g^{-1})^*$$

is again a polynomial coefficient differential operator. The action $Adg$ preserves addition and multiplication in $W_n$: it is an action by algebra automorphisms.

**Remark**: A differential operator $L$ is invariant for $Adg$ if and only if

$L$ commutes with the action of $g$ on $P(C^n)$.

Thus, if $G \subset GL_n$ is a group, the space

$$(W_n)^{AdG}$$

of $AdG$-invariant differential operators

is exactly the algebra of all differential operators that commute with $G$ acting on $P(C^n)$.
The Weyl Algebra, II: Conjugation by $GL_n$, cont.

More specifically $Ad g$ will preserve the spaces $X$ and $Y$ of multiplication by linear functions, and directional derivatives. We have

$$Y_n \cong \mathbb{C}^n, \quad \text{and} \quad X_n \cong (\mathbb{C}^n)^*$$

as $GL_n$ modules under $Ad$.

The infinitesimal action $ad = dAd$ of the Lie algebra $gl_n$ on $W_n$ derived from $Ad$ is just commutator with the operators $\alpha(T)$ described above:

if

$$ada(T)(L) = [\alpha(T), L] = \alpha(T)L - L\alpha(T),$$

for $T$ in $gl_n$, then

$$ada(T) = dAd(T),$$
The Weyl Algebra, III: Associated Graded Structure

The fact that the commutators of elements of \( W \) are scalar operators propagates through the spaces \( W^{(k)} \) of length \( k \) products, so that commutators are shorter than they might be:

\[
[W^{(k)}, W^{(\ell)}] \subset W^{(k+\ell-2)}.
\]

This means that, if we form the quotients

\[
\tilde{W}^{(k)} = \frac{W^{(k)}}{W^{(k-1)}},
\]

then the algebra

\[
\tilde{W} = \bigoplus_{k=0}^{\infty} \tilde{W}^{(k)}
\]

is commutative.

Remark: This is called the associated graded algebra.

In fact,

\[
\tilde{W} \simeq P(W)
\]

Tracing through all the identifications leads to:

The action \( Ad \) of \( GL_n \) on \( W_n \) factors to an action \( \tilde{Ad} \) on \( \tilde{W}_n \),

which in turn is isomorphic to the standard action of \( GL_n \) on \( P(C^n \oplus (C^n)^*) \).

In particular, this is a classical action.
The Weyl Algebra, IV: the metaplectic Lie algebra

The CCR imply that
\[ W_n^{(1)} = W_n + C = h_n \]
is a Lie algebra. It is 2-step nilpotent, with center = \( C \). It is the Heisenberg Lie algebra.

The CCR also imply that \( W^{(2)} \) is closed under commutator, i.e. is a Lie algebra. Further,
\[ [W^{(2)}, W^{(1)}] \subset W^{(1)}. \]
That is, \( W^{(2)} \) normalizes \( h_n \).

Refinement:
Let \( S^2(W) \subset W^{(2)} \) be the linear span of all symmetrized products (aka anticommutators)
\[ ww' + w'w, \quad \text{for } w, w' \text{ in } W. \]

Straightforward calculations with CCR show:

i) \( S^2(W_n) \) is a complement to \( h_n \) in \( W_n^{(2)} \): \( W_n^{(2)} = S^2(W_n) \oplus h_n. \)

ii) \( S^2(W_n) \) is a Lie algebra: \( [S^2(W_n), S^2(W_n)] \subset S^2(W_n). \)

iii) \( S^2(W_n) \) normalizes \( W_n \) inside \( h_n \): \( [S^2(W_n), W_n] \subset W_n. \)

Moreover,

iv) \( \text{ad}: S^2(W_n) \to End(W_n) \) given by taking brackets, identifies \( S^2(W_n) \) with the Lie algebra of \( Sp(W_n) \), where \( Sp(W_n) \) is the group of isometries of the commutator symplectic form \( << , >> \).
The Weyl Algebra, IV: the metaplectic Lie algebra, cont.

We call $S^2(W)$ the metaplectic Lie algebra. We write

$$S^2(W) = mp_{2n} = mp.$$ 

The decomposition $W = X \oplus Y$ implies

$$S^2(W) \simeq S^2(X) \oplus \{X,Y\} \oplus S^2(Y) = mp^{(2,0)} \oplus mp^{(1,1)} \oplus mp^{(0,2)}.$$ 

Concretely,

i) $mp^{(2,0)}$ consists of second order polynomials, and is spanned by $x_j x_k$.

ii) $mp^{(0,2)}$ consists of second order partial derivatives, and is spanned by $D_k D_j$.

iii) $mp^{(1,1)}$ is spanned by symmetrized products $\{D_k, x_j\} = D_k x_k + x_j D_k = 2x_j D_k + \delta_{jk}$.

The elements of $mp^{(1,1)}$ are almost the infinitesimal vector fields coming from $ad gl_n$, but there is an extra constant term.

They will have the same commutator action on $W_n$ as do the operators from $ad gl_n$. 


Stage II

Duality for Classical Actions

\( G = \) a classical group. Let \( \gamma : G \to GL_n \) be a classical action.

Then \( \text{Ad} \circ \gamma \) is the action by \( \gamma(G) \) on the differential operators by conjugation.

As noted, \( W_{n}^{\text{Ad} \gamma (G)} = \) algebra of differential operators commuting with the action of \( \gamma(G) \).

This will factor to an action \( \hat{\text{Ad}} \circ \gamma \) of \( G \) on \( \hat{W} \simeq P(W) \).

\( \gamma \) a classical action \( \implies \hat{\text{Ad}} \circ \gamma \) is also a classical action.

Hence Weyl’s FFT \( \implies \) the invariants for \( \hat{\text{Ad}} \circ \gamma(G) \) will be generated by the quadratic invariants.

This result pulls back to the Weyl algebra to give;

**Theorem:** If \( \gamma(G) \subset GL_n \) is a classical action, then \( W_{n}^{\text{Ad} \gamma (G)} \) is generated by the the Lie subalgebra

\[
g' = m_{p_{2n}}^{\text{Ad} \gamma (G)} \subset mp_{2n}
\]

that centralizes \( \gamma(G) \) inside \( mp_{2n} \).
Example: Spherical Harmonics

Let $G = O_{B_0} = O_n$, and $V \cong \mathbb{C}^n = \text{standard action for } G$.

The centralizing Lie algebra in $mp_{2n}$ is spanned by

\[ r^2 = \sum_{j=1}^{n} x_j^2, \quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \quad \tilde{E} = \sum_{j=1}^{n} \{D_j, x_j\} = 2 \left( \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \right) + n \]

Then

\[ \Delta : P^d(\mathbb{C}^n) \to P^{d-2}(\mathbb{C}^n). \]

\[ \ker \Delta \subset P^d(\mathbb{C}^n) = H^d(\mathbb{C}^n) = \text{harmonic polynomials of degree } d. \]

For this case, the duality theorem says:

**Theorem:** (Theory of Spherical Harmonics)

i) $H^d(\mathbb{C}^n)$ is an irreducible representation of $O_n$.

ii) Every polynomial $p$ in $P(\mathbb{C}^n)$ can be written uniquely as a sum

\[ p = \sum_{\ell \geq 0} q_\ell (r^2)^\ell \]

of products of harmonic polynomials with powers of $r^2$. 


Experience has shown:

Even if you are only interested in invariant functions,

you must also study how other functions transform under the action of a group.

This leads to the notion of representation of a group.

A representation \((\rho, V)\) of a group \(G\) on a vector space \(V\) is a group homomorphism

\[
\rho : G \rightarrow GL(V).
\]

Representation theory is a non-commutative analog of spectral theory.

Dictionary:

- invariant subspace \(\leftrightarrow\) subrepresentation
- eigenvector \(\leftrightarrow\) minimal invariant subspace \(=\) irreducible subrepresentation
- eigenvalues \(\leftrightarrow\) equivalence of subrepresentations
- eigenspace \(\leftrightarrow\) isotypic component

Main differences:

1. Irreducible representations are not simply specified by a number;
   
   they are a priori mysterious, and must be discovered.

2. Irreducible representations can have arbitrarily large dimension.

More precisely . . .
Key notions of Representation Theory

i) intertwining operator/\(G\)-morphism
ii) equivalence of representations.
iii) subrepresentation.
iv) irreducible representation.
v) (direct) sum of representations.

i): Given representations \((\rho, V)\) and \((\sigma, U)\) of \(G\),

an intertwining operator from \(V\) to \(U\) is a linear mapping

\[ L : V \rightarrow U \]

such that

\[ \sigma(g)L = L\rho(g), \text{ for all } g \text{ in } G. \]

ii) \((\rho, V)\) and \((\sigma, U)\) are equivalent

if there is an intertwining operator \(L : V \rightarrow U\)

that is an isomorphism of vector spaces.

iii): If there is an intertwining operator \(L : V \rightarrow U\)

that is one-to-one, then \(\rho\) is (equivalent to)

a subrepresentation of \(\sigma\).

If \(\{0\} \neq L(V) \neq U\), then \(\rho\) is a proper subrepresentation of \(\sigma\).

iv): If \(\sigma\) has no proper subrepresentations, it is irreducible.

(Equivalence classes of) irreducible representations of \(G\) = “atoms of symmetry” for \(G\) = \(\hat{G}\).

Goal 1: Describe the equivalence classes of irreducible representations = \(\hat{G}\).

Goal 2: Decompose a given representation into a sum of irreducible representations.
Tensor Products, Isotypic Components, Commutants, Burnside’s Theorem, I

Form the tensor product $U \otimes V$ of vector spaces $U$ and $V$.

Embed the algebra $\text{End}(U)$ of matrices on $U$ into $\text{End}(U \otimes V)$ by
$$\alpha(T)(u \otimes v) = T(u) \otimes v.$$  
Similarly, embed the algebra $\text{End}(V)$ of matrices on $V$ into $\text{End}(U \otimes V)$ by
$$\beta(S)(u \otimes v) = u \otimes S(v).$$

Given an (associative) algebra $A \subset \text{End}(X)$, the commutant of $A$ is
$$A' = \{ T \in \text{End}(X) : ST = TS, \text{ for all } S \in A \}$$

Pre-Burnside Theorem: The subalgebras $\alpha(\text{End}(U))$ and $\beta(\text{End}(V))$ are mutual commutants in $\text{End}(U \otimes V)$.

Also,
$$\alpha \otimes \beta : \text{End}(U) \otimes \text{End}(V) \longrightarrow \text{End}(U \otimes V)$$
is an isomorphism.
Tensor Products, Isotypic Components, Commutants, Burnside’s Theorem, II

Given:

\begin{align*}
\text{a representation } (\sigma, U) \text{ of } G \text{ on } U, \\
\text{and} \\
\text{irreducible } (\tau, W) \text{ in } \hat{G},
\end{align*}

the \textit{isotypic component} for \( \tau \) in \( \sigma \) is

the sum of all subrepresentations \( U_1 \subset U \) that are equivalent to \( \tau \).

For each such sub representation, \( U_1 \), there is an intertwining isomorphism

\[ L_1 : W \to U_1 \subset U. \]

Set

\[ U_\tau \text{ (aka } \sigma_\tau \text{)} = \tau\text{-isotypic component of } U \text{ (or of } \sigma). \]

\( H_{om_G}(W, U) = G\text{-intertwining operators from } W \text{ to } Z. \)

\textbf{Theorem:} The natural (evaluation) mapping

\[ W \otimes H_{om_G}(W, U) \to U_\tau \]

is an isomorphism.
Tensor Products, Isotypic Components, Commutants, Burnside’s Theorem, III

Given a group $G$ and a representation $\rho, V$ of $G$, one wants to find invariant subspaces. If a subspace $U$ is invariant under operators $\rho(g)$ and $\rho(g')$, it will also be invariant under the product $\rho(g)\rho(g') = \rho( gg')$. Thus, $U$ will be invariant under the algebra generated by taking all linear combinations of elements of $G$. So we subsume the problem of looking for sub representations for a group $G$ into the analogous problem for subalgebras $A$ of the $n \times n$ matrices.

To look for subspaces invariant under the algebra $A$, it can be helpful to find operators $S$ that commute with $A$. In particular, given such an operator $S$, all the eigenspaces of $S$ will be invariant under $A$. So we look for operators that commute with $A$. The collection of all such is another algebra, denoted $A'$, and called the commutant of $A$. If we can find $A'$, we could look for $(A')' = A''$, the double commutant of $A$. It is easy to check that $A \subset A''$. They might be equal, but they might not. This process could continue, but it turns out that $A''' = A'$ always. So if $A = A''$, the pairs of algebras $(A, A')$ form a pair of mutual commutants. There is a famous theorem of Burnside that describes what happens in this situation, if we assume that $A$ is semisimple, in the sense that $V$ is a sum of irreducible subrepresentations of $A$.

Burnside’s Theorem: If $A \subset M_n(C)$ is a semisimple subalgebra, then $A'$ is also semisimple. Moreover, there is a canonical decomposition

$$V = \sum_j U_j$$

such that:

i) Each $U_j$ is invariant under $A$ and under $A'$.

ii) The joint action of $A$ and $A'$ on $U_j$ is irreducible.

iii) Any two $A$-invariant irreducible subspaces of $U_j$ are equivalent as representations of $A$; same for $A'$.

iv) If $V_j$ is an $A$-irreducible subspace of $U_j$, and $V'_j$ is an $A'$-irreducible subspace of $U_j$, then $V_j \leftrightarrow V'_j$ defines a bijection of representations of $A$ and of $A'$.

v) $U_j \simeq V_j \otimes V'_j$.

Moral:
Finding the isotopic decomposition for $(\rho, V)$ and finding the commutant of $\rho(G)$ are closely related.
Let $G$ and $G'$ be two groups. Let $(\rho, V)$ and $(\xi, Y)$ be representations of $G$ and $G'$ respectively. Then $(\rho \otimes \xi, V \otimes Y)$ defined by

$$\rho \otimes \xi(g, g') (\vec{v} \otimes \vec{y}) = (\rho(g)(\vec{v}) \otimes \xi(g')(\vec{y}))$$

defines a representation $\rho \otimes \xi$ of $G \times G'$.

**Theorem:** The mapping

$$(\rho, \xi) \rightarrow \rho \otimes \xi$$

defines a bijection

$$\otimes : \hat{G} \times \hat{G}' \rightarrow (\hat{G} \times \hat{G}')$$
Theorem of the Highest Weight, I

$G = \text{classical group (over } \mathbb{C})$: \[ G \supset B \supset U; \quad B = A \cdot U, \]
where:

i) $B$ is a maximal connected solvable group.

ii) $U = \text{commutator subgroup of } B = \text{maximal unipotent subgroup of } G$.

iii) $A = \text{maximal diagonalizable subgroup of } B$.

A character of $A$ is a homomorphism $\psi : A \to \mathbb{C}^\times$.

If $(\rho, V)$ is a representation of $A$, and $\vec{v}$ in $V$ is an $A$-eigenvector, then

$$\rho(a)(\vec{v}) = \psi(a)\vec{v},$$

where $\psi$ is a character of $A$.

The collection of all characters of $A$ is $\hat{A}$: it is an abelian group.
Theorem of the Highest Weight, II

**Example:** $G = GL_n$: Then

- $B$ = group of (invertible) upper triangular matrices.
- $U$ = group of unipotent (all 1s on diagonal) upper triangular matrices.
- $A$ = group of (invertible) diagonal matrices:

  $$A = \left\{ a = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \right\}$$

Characters of $A$ have the form $\psi = \psi_m$, where $m = (m_1, m_2, m_3, \ldots m_n)$ is an $n$-tuple of integers, and $\psi_m: a \rightarrow \Pi_{j=1}^{n} a_j^{m_j}$

In particular, $A^m \simeq \mathbb{Z}^n$. 
Theorem of the Highest Weight, III

**Theorem**: Let \((\rho, V)\) be a representation of \(G\). Then

1) The space \(V^U\) of \(U\)-invariant vectors is non-zero.

2) \(V^U = \sum_{\psi \in \hat{A}} (V^U)^{(A,\psi)}\) can be decomposed into \(A\)-eigenspaces.

3) \(V\) irreducible \(\Rightarrow \dim V^U = 1\); and the \(A\)-character of \(V^U\) determines \(V\) up to equivalence.

Remark: The characters of \(A\) that appear as highest weight vectors is a subsemigroup of \(\hat{A}\), called the semigroup of *dominant* characters (or dominant weights). It is denoted \(\hat{A}^+\).

For \(G = GL_n\) the dominant weights are the \(\psi_\ell\) with descending entries: \(\ell_j \geq \ell_{j+1}\).
Theorem of the Highest Weight, IV, Diagram Notation

Given

\[ L = (\ell_1, \ell_2, \ell_3, \ldots, \ell_m) \]

= sequence of decreasing, non-negative integers.

For any \( n \geq m \), regard \( L \) as defining a dominant character of \( A_n \), by extending \( L \) with \( n - m \) zeroes.

Also, let \( L \) specify a diagram \( D_L = D \), consisting of left-justified rows of boxes, of lengths \( \ell_j \).

Then, for a diagram \( D \) consisting of \( m \) rows, we have for any \( n \geq m \), an associated irreducible representation \( \rho_n^D \) of \( GL_n \).

This is diagram notation.

There is a compatible* way of labeling \( \widehat{Sp}_{2n} \) and \( \widehat{O}_n \) with diagrams.

*For each irreducible representation \( \sigma \) of \( Sp_{2n} \), there is an irreducible representation \( \rho_{2n}^D \) such that \( D \) has at most \( n \) rows, and the highest weight vector of \( \rho_{2n}^D \), under the action of \( Sp_{2n} \), generates a representation equivalent to \( \sigma \). We write \( \sigma = \sigma^D \). A similar but more complicated procedure works for \( O_n \).
Stage II

Duality for Classical Actions

$G$ is a classical group. Let $\gamma: G \to GL_n$ be a classical action. Then $Ad \circ \gamma$ is the action by $\gamma(G)$ on the differential operators by conjugation.

As noted, $W_n^{Ad\gamma(G)} = \text{algebra of differential operators commuting with the action of } \gamma(G)$. This will factor to an action $\tilde{Ad} \circ \gamma$ of $G$ on $\tilde{W} \simeq P(W)$.

$\gamma$ a classical action $\implies \tilde{Ad} \circ \gamma$ is also a classical action. Hence Weyl’s FFT $\implies$ the invariants for $\tilde{Ad} \circ \gamma(G)$ will be generated by the quadratic invariants.

This result pulls back to the Weyl algebra to give:

**Theorem**: If $\gamma(G) \subset GL_n$ is a classical action, then $W_n^{Ad\gamma(G)}$ is generated by the Lie subalgebra $g' = mp_{2n}^{Ad\gamma(G)} \subset mp_{2n}$ that centralizes $\gamma(G)$ inside $mp_{2n}$.

Extension of the reasoning of Burnside’s Theorem $\implies$

**Corollary**:

$$P(C^n) \simeq \sum_D \sigma_D \otimes \tau_D,$$

where

$\sigma_D \in \hat{G}, \quad \tau_D \in \hat{g'},$

and

$\sigma_D \leftrightarrow \tau_D$

is one-to-one.
Duality for Classical Actions, cont.

Using the structure
\[ g' = g' \cap mp_{2n}^{(2,0)} \oplus g' \cap mp_{2n}^{(1,1)} \oplus g' \cap mp_{2n}^{(0,2)} \]
we get:

**Theorem:** Let \( H(G, \gamma) = \cap \Delta \) for \( \Delta \) in \( g'^{(0,2)} \).

Let \( J = P(C^n)^{\gamma(G)} \) be the algebra of invariants for \( \gamma(G) \) (which is generated by \( g'^{(2,0)} \)).

Let \( G'^{(1,1)} \) be the centralizer of \( \gamma(G) \) in \( GL_n \).

Then

i) (Harmonic Decomposition)

\[ P(C^n) = H(G, \gamma) \cdot J. \]

ii) (Harmonic Duality)

\[ H(G, \gamma) \simeq \sum D \sigma_D \otimes \rho_D, \]

where

\[ D = \text{as before,} \quad \text{and} \quad \rho_D \in \widehat{G'^{(1,1)}}. \]

Remark: \( G'^{(1,1)} \) is always a (product of) general linear group(s).
Dual Pairs

The duality theorem above relates representations of a classical group $G$ to representations of a commuting Lie algebra $g'$. In fact, the situation is in a suitable sense symmetric. The Lie algebra of $g$ is embedded in $mp_{2n}$, and each of $g$ and $g'$ is the centralizer of the other in $mp_{2n}$. We could also consider the group $Sp_{2n}$ whose Lie algebra is $mp_{2n}$, and look at the subgroups $G$ and $G'$ whose Lie algebras are $g$ and $g'$ respectively. Then either of $G$ or $G'$ is the centralizer of the other inside $Sp_{2n}$.

We call the pairs $(g, g')$ and $(G, G')$ of mutual centralizers dual pairs.

In what follows, we will talk about dual pairs in $Sp_{2n}$, but may actually be referring to some variant. All dual pairs in $Sp_{2n}$ arise via classical actions. Subject to a notion of irreducibility, there are two kinds of dual pairs in $Sp_{2n}(\mathbb{C})$:

$$(O_n, Sp_{2m}) \subset Sp_{2nm}, \quad (GL_n, GL_m) \subset Sp_{2nm}.$$ 

In $Sp_{2n}(\mathbb{R})$, one must consider various real forms of these pairs.
Stage III

See-Saw Pairs and Reciprocity

Suppose $H \subset G$, and $\gamma : G \to GL_n$ is a classical action such that $\gamma |_H$ is a classical action for $H$. Then $G = \gamma(G)$ and $H = \gamma(H)$ belong to dual pairs $(G, G')$ and $(H, H')$ in $Sp_{2n}$. We can put them in a diagram like this:

\[
\begin{array}{ccc}
G & \leftrightarrow & H' \\
\cup & \quad & \cup \\
H & \leftrightarrow & G'
\end{array}
\]

This diagram suggests the term see saw pair of dual pairs. This idea is due to S. Kudla.

**Reciprocity for See-Saw Pairs**: (Numerical version) If the duality correspondences for $(G, G')$ and $(H, H')$ are

\[
\sigma_D \leftrightarrow \tau_D \quad \text{and} \quad \tilde{\sigma}_E \leftrightarrow \tilde{\tau}_E,
\]

then the multiplicity with which $\tilde{\sigma}_E$ appears the restriction to $H$ of $\sigma_D$ equals the multiplicity with which $\tau_D$ appears in the restriction to $G'$ of $\tilde{\tau}_E$.

Remark: There is an elegant version of this theorem in terms of multigraded algebras.

**Examples**: If $H \subset G$ is a symmetric subgroup – the fixed points of an involution – then the restriction of a classical action of $G$ to $H$ will also be classical.
Symmetric Pairs and Bott Periodicity

There are 10 classes of symmetric pairs. These 10 can be organized into 5 see-saw pairs. These pairs fit nicely with Bott Periodicity in K-theory.

**The Real Periodicity Cycle**

\[
\begin{array}{c}
Sp_{2m}/GL_m \to \to Gl_n/O_n \\
\uparrow & \downarrow \\
((Sp_{2n} \times Sp_{2n})/Sp_{2n}) \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \to O_{p+q}/(O_p \times O_q) \\
\uparrow & \downarrow \\
Sp_{2(p+q)}/(S_{2p} \times Sp_{2q}) \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \to (O_n \times O_n)/O_n \\
\leftarrow & \leftarrow \\
GL_{2n}/Sp_{2n} \leftarrow \leftarrow \to O_{2m}/GL_m
\end{array}
\]

The symmetric pairs on each line correspond to each other via seesaw reciprocity.

**The Complex Periodicity Cycle**

\[
(GL_n \times GL_n)/GL_n \leftarrow \leftarrow \to GL_{p+q}/(GL_p \times GL_q)
\]
Reciprocity and Branching for Symmetric Pairs

For a subgroup $H$ of a group $G$, the decomposition of irreducible representations of $G$ into irreducible representations for $H$ is called a branching law. Branching laws are one of the basic problems of representation theory.

ia) Decomposition of tensor products $G \subset G \times G$, as the diagonal subgroup. For $GL_n$, the multiplicities are given by a combinatorial rule first described by Littlewood and Richardson, and are known as Littlewood-Richardson coefficients.

ib) The Littlewood-Richardson rule can be proved using representation theory and reciprocity between $(GL_n \times GL_n, GL_n)$ and $(GL_{p+q}, GL_p \times GL_q)$. (First representation-theory based proof.)

ii) By using appropriate reciprocity laws, all branching rules for classical symmetric pairs $(G, H)$ can be described in terms of Littlewood-Richardson coefficients.
Stage IV

Fine Structure of the Harmonic Decomposition

A key part of the theory of spherical harmonics is the uniqueness of the decomposition of a general polynomial into a sum of harmonics times powers of $r^2$. In general, the duality theorem guarantees that any polynomial is a sum of products of $(G, \gamma)$ harmonics with $\gamma(G)$ invariants. However, in general, this decomposition is not unique. It is unique when $g'$, or equivalently, the number of copies of the standard module of $G$, is relatively small. This is known as the stable range. However, as $g'$ grows for a fixed $G$, eventually uniqueness fails. This means that the mapping

$$\mu : H(G, \gamma) \otimes J \rightarrow P(\mathbb{C}^n)$$

induced by multiplication in $P(\mathbb{C}^n)$ has a non-trivial kernel. It is then a natural question, to describe the kernel.

This question has proved fairly resistant. It seems to require understanding of the structure of the individual factors $H(G, \gamma)$ and $J$. Of course, $J$ is an algebra. Let $I(J)$ denote the ideal in $P(\mathbb{C}^n)$ generated by $J(2)$, the quadratic generators of $J$. The harmonic decomposition says that the quotient mapping

$$H(G, \gamma) \rightarrow P(\mathbb{C}^n)/I(J)$$

is a linear isomorphism. Thus, we may think of $H(G, \gamma)$ as an algebra by pullback of structure from $P(\mathbb{C}^n)/I(J)$.

There has been recent progress in understanding the structure of $H(G, \gamma)$ and of $J$.

**Theorem** $H(G, \gamma)$ and $J$ (nearly) have flat deformations to Hibi rings. In particular, each of $H(G, \gamma)$ and $J$ has a standard monomial theory.

**Corollary:** $H(G, \gamma) \otimes J$ has a standard monomial theory.
What is Standard Monomial Theory?

Abstract Rough Version

Standard monomial theory is the expression of a ring as an “almost direct sum” of polynomial rings.

I.e, the collections of all monomials in certain subsets of generators form a basis for the ring.
Example: Hodge’s standard monomial theory for $GL_n$ (modern version), I

The Flag Algebra.

$G =$ reductive group (over $\mathbb{C}$).

$U = U_G =$ maximal unipotent subgroup,

$A_G = A =$ maximal torus normalizing $U$.

$\hat{A} =$ group of (regular) characters of $A$

$\hat{A}^+ =$ semigroup of dominant characters of $A$.

The Theorem of the Highest Weight, combined with Frobenius Reciprocity, implies that:

$$R(G/U) = \text{(ring of) regular functions on } G/U$$

\[ \cong \sum_{\rho \in \hat{G}} V_{\rho} \]

\[ = \sum_{\psi \in \hat{A}^+} V_{\psi}. \]

The $V_{\psi}$ are the eigenspaces for the right action of $A$.

Thus, $R(G/U) = \hat{A}$-graded algebra.
Example: Hodge’s standard monomial theory for $GL_n$ (modern version), II

Consider the subset $\Gamma_n \subset \mathbb{Z}^2$, with the standard partial order:

\[
\Gamma_6 =
\begin{array}{ccccccc}
& & & & & & * \\
* & & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

Then $R^+ (GL_n/U_n)$ has a flat deformation to the semigroup ring of all

order-preserving

non-negative

integer-valued

functions on $\Gamma_n$

Aka, the Hibi cone on $\Gamma_n$. 
Hibi Cones

Let $\Gamma$ be a partially ordered set (poset). ($\succeq = \text{the order relation}$.)

$R_\Gamma = \text{real-valued functions on } \Gamma.$

$(R^+_\Gamma) = \text{non-negative real-valued functions on } \Gamma.$

$R^{\Gamma,\succeq} = \text{order-preserving real-valued functions on } \Gamma.$

$(R^+_\Gamma)^{\Gamma,\succeq} = R^{\Gamma,\succeq} \cap (R^+_\Gamma).$

$Z_\Gamma = \text{integer-valued functions on } \Gamma.$

$(Z^+_\Gamma) = Z \cap (R^+_\Gamma).$

$Z^{\Gamma,\succeq} = Z \cap R^{\Gamma,\succeq};$  $(Z^+_\Gamma)^{\Gamma,\succeq} = Z \cap (R^+_\Gamma)^{\Gamma,\succeq}.$

$(Z^+_\Gamma)^{\Gamma,\succeq}$ is a lattice cone.

The Hibi ring attached to $\Gamma$ is

$C((Z^+_\Gamma)^{\Gamma,\succeq}),$ the semigroup ring of $(Z^+_\Gamma)^{\Gamma,\succeq}.$

The structure of $(Z^+_\Gamma)^{\Gamma,\succeq}$ affords an explicit description

of the generators and relations for the associated Hibi ring.
Structure of $(\mathbb{Z}^+)_{\Gamma, \succeq}$.

Recall $\Gamma_C^+$ ( = Hibi cone for a total ordering.)

Let $TO(\Gamma, \succeq) =$ collection of total orderings of $\Gamma$ compatible with (i.e. that extend) the partial order $\succeq$.

Then

i) $(\mathbb{Z}^+)_{\Gamma, \succeq} = \bigcup_{\succeq_j \in TO(\Gamma, \succeq)} (\mathbb{Z}^+)_{\Gamma, \succeq_j}$.

ii) For each $\succeq_j$ in $TO(\Gamma, \succeq)$, $(\mathbb{Z}^+)_{\Gamma, \succeq_j} \simeq C_{D_n}$ (with $n = \#(\Gamma)$).

Given poset $\Gamma$, let $INC(\Gamma) =$ collection of all increasing subsets of $\Gamma$.

$INC(\Gamma)$ is closed under taking unions and intersections.

$INC(\Gamma)$ is ordered by inclusion. $\Gamma \hookrightarrow INC(\Gamma)$.

Note: $TO(\Gamma, \succeq) \leftrightarrow$ maximal chains $\subseteq INC(\Gamma)$.

Then

i) $C(\mathbb{Z}^+\Gamma, \succeq)$ is generated by the characteristic functions $\chi_B$, for increasing subsets $B \subset \Gamma$.

ii) Defining relations are $\chi_B + \chi_{B'} = \chi_{B \cup B'} + \chi_{B \cap B'}$.

iii) For each $\succeq_j \in TO(\Gamma), \quad C((\mathbb{Z}^+\Gamma, \succeq_j))$ is a polynomial subring of $C(\mathbb{Z}^+\Gamma, \succeq)$.

iv) $C((\mathbb{Z}^+\Gamma, \succeq_j))$ is generated by $\chi_B$, for $B$ in the maximal chain of $\succeq_j$ in $INC(\Gamma)$.

v) $C(\mathbb{Z}^+\Gamma, \succeq) = \text{almost direct sum of } C((\mathbb{Z}^+\Gamma, \succeq_j))$. 
The kernel of $m$

(case of $GL_n$ on $V^p \oplus (V^*)^q$).

**Theorem:** The obvious elements generate $\ker m$.

More precisely:

Consider the matrix

$$
\begin{bmatrix}
  r_{11} & r_{21} & r_{31} & \cdots & r_{p1} & y_{11} & y_{21} & y_{31} & \cdots & y_{n1} \\
  r_{12} & r_{22} & r_{23} & \cdots & r_{2p} & y_{12} & y_{22} & y_{32} & \cdots & y_{n2} \\
  r_{13} & r_{23} & r_{33} & \cdots & r_{p3} & y_{21} & y_{21} & y_{31} & \cdots & y_{n1} \\
  & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  r_{1q} & r_{2q} & r_{3q} & \cdots & r_{pq} & y_{1q} & y_{2q} & y_{3q} & \cdots & y_{nq} \\
  x_{11} & x_{12} & x_{13} & \cdots & x_{1p} & 0 & 0 & 0 & \cdots & 0 \\
  x_{21} & x_{22} & x_{23} & \cdots & x_{2p} & 0 & 0 & 0 & \cdots & 0 \\
  x_{31} & x_{32} & x_{33} & \cdots & x_{3p} & 0 & 0 & 0 & \cdots & 0 \\
  & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  x_{n1} & x_{n2} & x_{n3} & \cdots & x_{np} & 0 & 0 & 0 & \cdots & 0 
\end{bmatrix}
$$

**Lemma:** This matrix has rank $n$.

**Theorem:** (w/ Soo Teck Lee) The kernel of $m$ is generated by the determinants of the $(n+1) \times (n+1)$ submatrices.