Hyperdeterminants from E8

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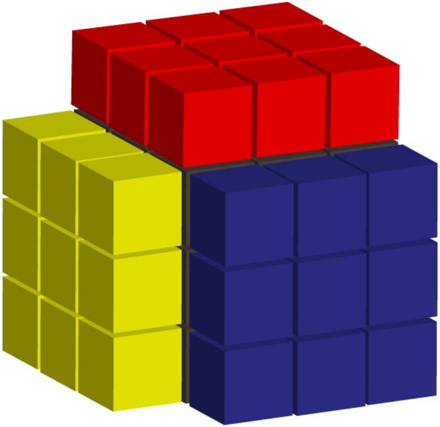
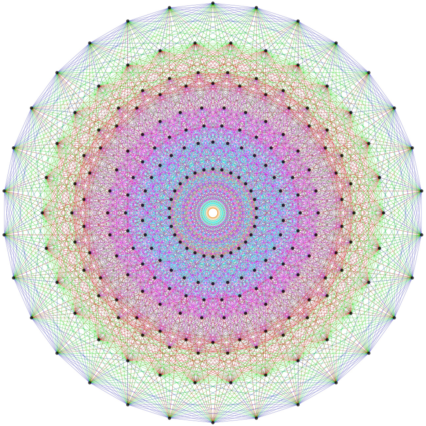
Overview: Hyperdeterminants from E8 (with Holweck)

Use invariant theory of complex semi-simple Lie algebras (like $\mathfrak{e}_8$) to give practical computations that separate entanglement classes for various formats of tensors, like $\mathbb{C}^{2\times 2\times 2\times 2}$, $\mathbb{C}^{3\times 3\times 3}$, $\bigwedge^3\mathbb{C}^9$, $\bigwedge^4\mathbb{C}^8$.

(qubits, qutrits, fermionic systems)





Advertisement: Learning Algebraic Models of Entanglement (with Jaffali)

Use artificial neural networks (hybrid with ReLU and power activation functions) and Algebraic Geometry as well as symmetry enhanced methods to detect different types of entanglement for up to 5 qubit ($\mathbb{C}^{2\times2\times2\times2\times2}$) and 3 qutrit systems ($\mathbb{C}^{3\times3\times3}$).

Tensor size	Architecture	Training acc.	Validation acc.	Testing acc.	Loss
$2\times2\times2$	(100,50,25,16,1)	93.44%	92.53%	92.74%	0.1629
$2^{\times4}$	(200,100,50,16,1)	99.50%	95.95%	95.94%	0.01791
$2^{\times5}$	(100,50,25,16,1)	99.95%	98.74%	98.83%	0.001533
$3\times3\times3$	(100,50,25,16,1)	98.18%	96.78%	96.83%	0.04770

TABLE 5. LeakyReLU network architectures and accuracies for each tensor size, for degenerate and non-degenerate states classification.

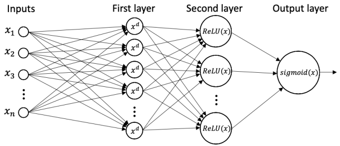


FIGURE 8. Representation of a hybrid network for learning a homogeneous polynomial equation of degree d in n variables and coefficients in $\mathbb{R}$.

Entangled and Degenerate states

Definition

Hilbert space $\mathcal{H} = \mathbb{C}^n$ (state space for a particle).
The state of an m-particle ensemble $\Phi \in \mathcal{H}^{\otimes m}$ is *unentangled* if $\Phi = \varphi_1 \otimes \cdots \otimes \varphi_n$ for some $\varphi_i \in \mathcal{H}$.
Unentangled ensambles satisfy *independence*: $P(\Phi_i = a \mid \Phi_j = b) = P(\Phi_i = a)P(\Phi_j = b)$.

Example (Entangled States)

Take $\Phi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Measure one particle at a time (separate experiments):
$P(\varphi_1 = |0\rangle) = P(\varphi_1 = |1\rangle) = 50\%$, $\quad P(\varphi_2 = |0\rangle) = P(\varphi_2 = |1\rangle) = 50\%$.
But, $P(\varphi_2 = |0\rangle \mid \varphi_1 = |0\rangle) = 100\%$ and $P(\varphi_2 = |0\rangle \mid \varphi_1 = |1\rangle) = 0\%$.
The independence condition $P(\varphi_2 = b \mid \varphi_1 = a) = P(\varphi_2 = b)P(\varphi_1 = a)$ fails.

Remark

Invariants distinguish states: For 2-particle systems matrix rank completely classifies entanglement type.
$|00\rangle \to \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ *and* $\det \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right) = 0 \Rightarrow$ *unentangled.*
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \to \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ *and* $\det \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) = \frac{1}{2} \Rightarrow$ *entangled.*

Determinant: Dual to rank 1 matrices (unentangled states)

A matrix has rank 1 if, up to Gaussian elimination it is of the form:

$$A = \begin{pmatrix} * & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad = \mathit{column} \cdot \mathit{row}$$

Dually, a matrix is singular if up to Gaussian elimination it is of the form:

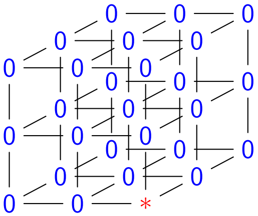
$$A^\vee = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & * & * & \dots & * \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & * & * & \dots & * \end{pmatrix} \qquad = \mathit{annihilates\ row\ and\ column}$$

The set of all singular matrices is defined by the vanishing of a determinant.

Hyperdeterminant: Dual to rank 1 tensors (unentangled states)

Dual to rank 1 tensors are singular tensors, defined by the vanishing of a hyperdeterminant [GKZ].

Rank 1 | Tangent | Hyperplanes $\perp$



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$\mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$

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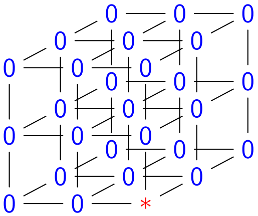

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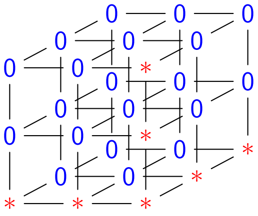
$\mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$

Tangent



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$T_p \, \mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$

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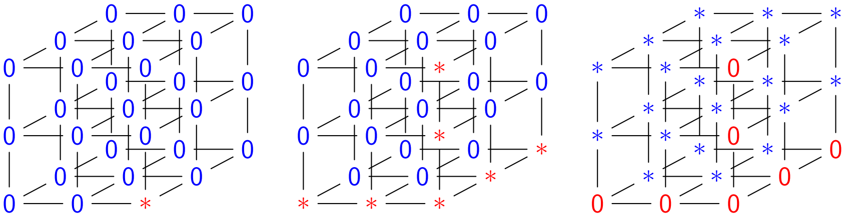

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$\mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$ | $T_p\,\mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)$ | $\mathrm{Seg}(\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n)^\vee = \mathcal{V}(Det)$

Random arrays of this type and change coordinates – get points with vanishing hyperdeterminant.

Singular Tensors

- Classical Algebraic Geometry: Given a variety $X \subset \mathbb{P}^n$, describe its projective dual $X^\vee \subset (\mathbb{P}^n)^*$.
  The (closure of) the set of hyperplanes tangent to X at smooth points.
- Matrices: $X = \mathrm{Seg}(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$ rank-1 matrices
  $X^\vee = \mathrm{Seg}(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})^\vee$ co-rank-1 matrices $X^\vee = \mathcal{V}(det)$
- Tensors: $X = \mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, $X^\vee = \mathcal{V}(Det)$

$$\begin{tikzcd}
 & \frac{1}{\sqrt{2}}(|111\rangle + |222\rangle) \arrow[d] & \\
 & \frac{1}{\sqrt{3}}(|112\rangle + |121\rangle + |211\rangle) \arrow[dl] \arrow[d] \arrow[dr] & \\
\frac{1}{\sqrt{2}}(|112\rangle + |121\rangle) \arrow[dr] & \frac{1}{\sqrt{2}}(|112\rangle + |211\rangle) \arrow[d] & \frac{1}{\sqrt{2}}(|121\rangle + |211\rangle) \arrow[dl] \\
 & |111\rangle \arrow[d] & \\
 & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{C}^{2\times 2\times 2} \arrow[d] & \\
 & X^\vee = \tau(X) \arrow[dl] \arrow[d] \arrow[dr] & \\
\tau_{23} \arrow[dr] & \tau_{13} \arrow[d] & \tau_{12} \arrow[dl] \\
 & X \arrow[d] & \\
 & \emptyset &
\end{tikzcd}$$

- Quantum Information: Invariants for singular tensors measure entanglement.

Warmup: One construction to detect degeneracy for 3 different types

$\bigwedge^3\mathbb{C}^6$ (3 fermionic particles with 6 states each), $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$ (3 qubits), $S^3\mathbb{C}^2$ (3 binary bosons)
Invariant rings $\mathbb{C}[f_4] = \mathbb{C}[\bigwedge^3\mathbb{C}^6]^{SL(6)} \cong \mathbb{C}[(\mathbb{C}^2)^{\otimes 3}]^{SL(2)^{\times 3}} \cong \mathbb{C}[S^3\mathbb{C}^2]^{SL(2)}$.

Katanova-like method for computing f_4:

Given $T \in \bigwedge^3\mathbb{C}^6$ get natural maps:

$$\bigwedge^1\mathbb{C}^6 \xrightarrow{\wedge T} \bigwedge^4\mathbb{C}^6 \quad \text{and} \quad \bigwedge^2\mathbb{C}^6 \xrightarrow{\wedge T} \bigwedge^5\mathbb{C}^6$$

Given $0 \neq \Omega \in \bigwedge^6\mathbb{C}^6$ get natural isomorphisms:

$$\bigwedge^5\mathbb{C}^6 \xrightarrow{\lrcorner\Omega} \bigwedge^1\mathbb{C}^6 \quad \text{and} \quad \bigwedge^4\mathbb{C}^6 \xrightarrow{\lrcorner\Omega} \bigwedge^2\mathbb{C}^6$$

Get linear map depending quadratically on T.

$$\bigwedge^1\mathbb{C}^6 \xrightarrow{\Omega T \Omega T} \bigwedge^1\mathbb{C}^6$$

Take power trace to get invariants:

$$\mathrm{trace}(\Omega T \Omega T) = 0 \quad \mathrm{trace}((\Omega T \Omega T)^2) = f_4$$

Note that $\mathbb{C}^2\oplus\mathbb{C}^2\oplus\mathbb{C}^2 = \mathbb{C}^6$ so

$$\bigwedge^3\left(\mathbb{C}^2\oplus\mathbb{C}^2\oplus\mathbb{C}^2\right) \supset \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2 \supset S^3\mathbb{C}^2$$

One checks that the restrictions are non-zero:

$$(f_4)_{|(\mathbb{C}^2)^{\otimes 3}} = \Delta_{222}$$
$$(f_4)_{|S^3\mathbb{C}^2} = \mathrm{disc}_3$$

Compute invariants for all 3 by restricting a power trace of a matrix to the appropriate tensor format. Easy-Peezy!

Sparsifying invariants is projection

Elementary School: $f = ax^2 + 2bxy + cy^2$ (for a, b, c parameters, x, y variables).

The discriminant $\Delta = ac - b^2 \in \mathbb{C}[a, b, c]$ vanishes when f has a double root.

Project to the subspace of parameters where $b = 0$ (diagonal matrices).

Now $\Delta_{|b=0} = ac \in \mathbb{C}[a, c]$ vanishes when $f_{|b=0}$ is the square of a variable (rank-1 diagonal).

Apply Schläfli

The top orbit is non-degenerate:

$$S = |111\rangle + |222\rangle \quad \to \quad S(x) = x_1\,|11\rangle + x_2\,|22\rangle = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$$

$$\det(S(x)) = x_1 x_2 \quad \to \operatorname{disc}\det(S(x)) = 1^2 - 4(0)(0) = 1 \neq 0$$

The next orbit is degenerate:

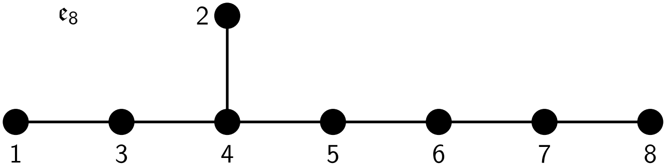
$$W = |112\rangle + |121\rangle + |211\rangle \quad \to \quad W(x) = x_1\,|12\rangle + x_1\,|21\rangle + x_2\,|11\rangle = \begin{pmatrix} x_2 & x_1 \\ x_1 & 0 \end{pmatrix}$$

$$\det(W(x)) = -x_1^2 \quad \to \operatorname{disc}\det(W(x)) = 0^2 - 4(-1)(0) = 0$$

Works for several cases such as $2 \times 2 \times 2 \times 2$ and $3 \times 3 \times 3$, (see GKZ, Weyman-Zelevinsky) but until now, no known effective method for $\bigwedge^3 \mathbb{C}^9$, $\bigwedge^4 \mathbb{C}^8$, for example.

Some groups and some equivariant projections

$\mathfrak{e}_8$



Do you see $A8$? $A2A2A2$? Do you see $E7$? $A7$? Do you see $E6$? $A6$?

Vinberg: $\mathbb{Z}_3$-grading: $\mathfrak{e}_8 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \bigwedge^6 V \oplus \mathfrak{sl}(V) \oplus \bigwedge^3 V$ for $V = \mathbb{C}^9$

Get $\mathfrak{sl}_9$-equivariant projection: $\mathfrak{e}_8 \to \bigwedge^3 V$.

Given G-module $V = A \oplus B$, and H-module A, with $H < G$.
Ask about H-equivariant projections of G invariants on V to H invariants on A.
Sometimes get nice divisibility results.

Invariants

- The *Jordan-Chevelley decomposition* of a Lie algebra: $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$ (semi-simple plus nilpotent).
  Continuous invariants are determined by their value on the semi-simple part (of the given tensor).

- Chevalley Restriction Theorem: $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{h}]^W$, where $\mathfrak{g}$ is a complex semi-simple Lie algebra associated with the Lie group G, $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra and W is the Weyl group.

- Tevelev: For G complex semi-simple, and X_G^{ad} the adjoint orbit. Then $X^\vee$ is a hypersurface defined by F, and the restriction to semi-simple elements is

$$F_{\mathfrak{s}} = \prod_{\alpha \in R} \alpha.$$

(High-powered version of "Determinant is product of eigenvalues")

Vinberg: $\mathbb{Z}_3$-grading: $\mathfrak{e}_8 = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \bigwedge^3 V^* \oplus \mathfrak{sl}(V) \oplus \bigwedge^3 V \quad$ for $\quad V = \mathbb{C}^9$
Root system for $E8$:

$$\Sigma = \{\varepsilon_i - \varepsilon_j, \pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)\} \quad (i, j, k \text{ distinct}).$$

Cartan subalgebra (semi-simple elements): Typical choice $\mathfrak{h} \subset \mathfrak{sl}(V)$. Another choice: $\mathfrak{s} = \mathfrak{s}_{-1} \oplus \mathfrak{s}_1$ with each $\mathfrak{s}_i \subset \mathfrak{g}_i$ a 4-dimensional space of semi-simple elements in $\bigwedge^3 V$. $\mathfrak{s}_1$ spanned by:

$$\begin{aligned} p_1 &= e_{123} + e_{456} + e_{789}, & p_2 &= e_{147} + e_{258} + e_{369}, \\ p_3 &= e_{159} + e_{267} + e_{348}, & p_4 &= e_{168} + e_{249} + e_{357}, \end{aligned}$$

Tevelev: $\Delta_{E8} = \prod_{\alpha \in \Sigma} \alpha$

Have $\mathfrak{sl}_9$-equivariant projection: $\mathfrak{e}_8 \to \bigwedge^3 V$. Restrict Δ_{E8} to semi-simple parts.
Change coordinates and express Σ in terms of $\mathfrak{s}$, then restrict to $\mathfrak{s}_1$:
Generic semi-simple form:

$$p = z_1 p_1 + z_2 p_2 + z_3 p_3 + z_4 p_4$$

Jordan Decomposition: How to make a large invariant fit on a single slide

A tensor is considered *nilpotent* (with respect to a Lie algebra $\mathfrak{g}$ action) if 0 is contained in its orbit.
A tensor is considered *semi-simple* if it has a closed orbit.
Jordan decomposition for vectors $u = p + e$ with p semi-simple, e nilpotent, and $p \wedge e = 0$.

Proposition

If f is a continuous invariant, then $f(u) = f(p)$.

We have the following expression (with $\omega^3 = 1$): $[(\Delta_{E_8})_{|\mathfrak{s}_1}] = [h^6]$, with $h =$

$$
\begin{aligned}
&z_4(z_1-z_2+z_3)(z_1+\omega z_2+z_3)(z_1-z_2-\omega z_3)(z_1+\overline{\omega} z_2+z_3)(z_1-z_2-\overline{\omega} z_3)(z_1+\omega z_2-\omega z_3)(z_1+\omega z_2-\overline{\omega} z_3)(z_1+\overline{\omega} z_2-\omega z_3)(z_1+\overline{\omega} z_2-\overline{\omega} z_3)\\
&z_3(z_1+z_2-z_4)(z_1-\omega z_2-z_4)(z_1+z_2+\omega z_4)(z_1-\overline{\omega} z_2-z_4)(z_1+z_2+\overline{\omega} z_4)(z_1-\omega z_2+\omega z_4)(z_1-\overline{\omega} z_2+\omega z_4)(z_1-\omega z_2+\overline{\omega} z_4)(z_1-\overline{\omega} z_2+\overline{\omega} z_4)\\
&z_2(z_1-z_3+z_4)(z_1+\omega z_3+z_4)(z_1-z_3-\omega z_4)(z_1+\overline{\omega} z_3+z_4)(z_1-z_3-\overline{\omega} z_4)(z_1+\omega z_3-\omega z_4)(z_1+\omega z_3-\overline{\omega} z_4)(z_1+\overline{\omega} z_3-\omega z_4)(z_1+\overline{\omega} z_3-\overline{\omega} z_4)\\
&z_1(z_2+z_3+z_4)(z_2-\omega z_3+z_4)(z_2+z_3-\omega z_4)(z_2-\overline{\omega} z_3+z_4)(z_2+z_3-\overline{\omega} z_4)(z_2-\omega z_3-\omega z_4)(z_2-\overline{\omega} z_3-\omega z_4)(z_2-\omega z_3-\overline{\omega} z_4)(z_2-\overline{\omega} z_3-\overline{\omega} z_4).
\end{aligned}
$$

Note $[\Delta^2_{\mathrm{Gr}(3,9)}] = [\pi_{\bigwedge^3 \mathbb{C}^9} \Delta_{E_8}]$ is equivalent to knowing that $[(\Delta_{E_8})_{|\mathfrak{s}_1}] = [h^6]$ and $[(\Delta_{\mathrm{Gr}(3,9)})_{|\mathfrak{s}_1}] = [h^3]$.
We have several other similar formulas for other discriminants.

Restrictions of the E8-discriminant and divisibility

Restrict the $E8$ discriminant to many other H-invariant subspaces:

$$\begin{tikzcd}
\Delta_{E_8} \arrow[r] \arrow[d, "\pi_{((\wedge^3\mathbb{C}^9)^*)^\perp}"] & \Delta_{E_7}|\pi(\Delta_{E_8}) \arrow[r] \arrow[d, "\pi_{((\wedge^4\mathbb{C}^8)^*)^\perp}"] & \Delta_{E_6}|\pi(\Delta_{E_7}) \arrow[r] \arrow[d, "\pi_{(\mathbb{C}^3)^{\otimes 3}}"] & \Delta_{SO(8)}|\pi(\Delta_{E_6}) \arrow[d, "\pi_{((\mathbb{C}^2)^{\otimes 4*})^\perp}"] \\
\Delta^2_{\mathrm{Gr}(3,9)}|\pi(\Delta_{E_8}) \arrow[d, "\pi_{((\mathbb{C}^3)^{\otimes 3*})^\perp}"] & \Delta_{\mathrm{Gr}(4,8)}|\pi(\Delta_{E_7}) \arrow[d, "\pi_{((\mathbb{C}^2)^{\otimes 4*})^\perp}"] & \Delta^2_{(\mathbb{P}^2)^{\times 3}}|\pi(\Delta_{E_6}) \arrow[d, "\pi_{(Sym^3(\mathbb{C}^3)^*)^\perp}"] & \Delta_{(\mathbb{P}^1)^{\times 4}}|\pi(\Delta_{SO(8)}) \arrow[d, "\pi_{(Sym^4(\mathbb{C}^2)^*)^\perp}"] \\
\Delta_{(\mathbb{P}^2)^{\times 3}}|\pi(\Delta_{\mathrm{Gr}(3,9)}) \arrow[d, "\pi_{(Sym^3(\mathbb{C}^3)^*)^\perp}"] & \Delta_{(\mathbb{P}^1)^{\times 4}}|\pi(\Delta_{\mathrm{Gr}(4,8)}) \arrow[d, "\pi_{(Sym^4(\mathbb{C}^2)^*)^\perp}"] & \Delta_{v_3(\mathbb{P}^2)}|\pi(\Delta_{(\mathbb{P}^2)^{\times 3}}) & \Delta_{v_4(\mathbb{P}^1)}|\pi(\Delta_{(\mathbb{P}^1)^{\times 4}}) \\
\Delta_{v_3(\mathbb{P}^2)}|\pi(\Delta_{(\mathbb{P}^2)^{\times 3}}) & \Delta_{v_4(\mathbb{P}^1)}|\pi(\Delta_{(\mathbb{P}^1)^{\times 4}}) & &
\end{tikzcd}$$

Get the $E7$, $E6$ and $SO(8)$ discriminants, the $2 \times 2 \times 2 \times 2$ and $3 \times 3 \times 3$ hyperdeterminants, the $Gr(3,9)$ and $Gr(4,8)$ discriminants, and more.

Interpolation in monomials of lower degree invariants

Katanova: There exists an 84×84 matrix C depending cubicly on $T \in \bigwedge^3 \mathbb{C}^9$ so that:

$$\mathrm{trace}\, C^m = f_{3m}$$

and the invariant ring is $\mathbb{C}[f_{12}, f_{18}, f_{24}, f_{30}] = \mathbb{C}[\bigwedge^3 \mathbb{C}^9]^{\mathsf{SL}(9)}$.
Obtain an evaluation method even when the hyperdeterminant is very large.
$\Delta_{\mathsf{Gr}(3,9)} \in \mathbb{C}[\bigwedge^3 \mathbb{C}^9]^{\mathsf{SL}(9)}$ has degree 120 on 84 variables. Might expect $\sim 10^{58}$ monomials.
However, there are 28 monomials in $f_{12}, f_{18}, f_{24}, f_{30}$ of degree 120.
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$$\begin{aligned}
&f_{12}^{10}-\frac{188875}{1526823}f_{12}^{8}f_{24}-\frac{449402187651722270463}{2232199994248855116}f_{12}^{7}f_{18}^{2}+\frac{522717082571600510}{5022449987059924011}f_{12}^{6}f_{18}f_{30}+\frac{156259946875}{27974261679948}f_{12}^{6}f_{24}^{2}+\frac{20955843759677134000}{15067349961179772033}f_{12}^{5}f_{18}^{2}f_{24}\\
&+\frac{113325967730636958495085217}{1009180965699898771226274}f_{12}^{4}f_{18}^{4}-\frac{8007699664851700}{45202049883539316099}f_{12}^{5}f_{30}^{2}-\frac{951594557840795000}{135606149650617948297}f_{12}^{4}f_{18}f_{24}f_{30}-\frac{37339826093750}{327991224631970313}f_{12}^{4}f_{24}^{3}\\
&-\frac{4631798176278228432974860}{4541314345649544470518233}f_{12}^{3}f_{18}^{3}f_{30}-\frac{43381098724294271875}{2440910693711123069346}f_{12}^{3}f_{18}^{2}f_{24}^{2}-\frac{48098757899275092625}{15067349961179772033}f_{12}^{2}f_{18}^{4}f_{24}-\frac{11518845901768651039}{329340982758027804}f_{12}f_{18}^{6}\\
&+\frac{1392403335812500}{135606149650617948297}f_{12}^{3}f_{24}f_{30}^{2}+\frac{6686357462527147925300}{1513771448549848156839411}f_{12}^{2}f_{18}^{2}f_{30}^{2}+\frac{140973248590625000}{1220455346855561534673}f_{12}^{2}f_{18}f_{24}^{2}f_{30}+\frac{351718750000}{327991224631970313}f_{12}^{2}f_{24}^{4}\\
&+\frac{2133816827644645000}{135606149650617948297}f_{12}f_{18}^{3}f_{24}f_{30}-\frac{198339133437500}{741017211205562559}f_{12}f_{18}^{2}f_{24}^{3}+\frac{45691574382263590}{741017211205562559}f_{18}^{5}f_{30}-\frac{32778366465625}{48591292538069676}f_{18}^{4}f_{24}^{2}\\
&-\frac{14445540571041712000}{1513771448549848156839411}f_{12}f_{18}f_{30}^{3}-\frac{216716472500000}{1220455346855561534673}f_{12}f_{24}^{2}f_{30}^{2}-\frac{2371961791512500}{135606149650617948297}f_{18}^{2}f_{24}f_{30}^{2}+\frac{10890275000000}{20007464702550189093}f_{18}f_{24}^{3}f_{30}\\
&-\frac{1250000000}{327991224631970313}f_{24}^{5}+\frac{34328756109890000}{4541314345649544470518233}f_{30}^{4}.
\end{aligned}$$

Computing in the invariant ring $\mathbb{C}[\bigwedge^4\mathbb{C}^8]^{SL(8)} = \mathbb{C}[f_2, f_6, f_8, f_{10}, f_{12}, f_{14}, f_{18}]$

Given $T \in \bigwedge^4\mathbb{C}^8$ get mappings $\wedge T\colon \bigwedge^2\mathbb{C}^8 \to \bigwedge^6\mathbb{C}^8$ and $\lrcorner T\colon \bigwedge^6\mathbb{C}^8 \to \bigwedge^2\mathbb{C}^8$.
Compose $A := (\lrcorner T)(\wedge T)$ get a matrix $\bigwedge^2\mathbb{C}^8 \to \bigwedge^2\mathbb{C}^8$ depending quadratically on T.
Check: traces of the powers of appropriate degrees produce the fundamental invariants:

$$f_{2n} = tr(A)^n, \quad \text{for} \quad n = 1, 3, 4, 5, 6, 7, 9.$$

This can be done in a few lines in Macaulay2, for instance.

```
mysubsets = (a,b) -> apply(subsets(a,b), xx-> xx+apply(b, i-> 1));
R = QQ[e_1..e_8,SkewCommutative=> true];
S = QQ[apply(mysubsets(8,4), i-> x_i)];
RS = R**S;
myRules = sum(mysubsets(8,4),  I -> e_(I_0)*e_(I_1)*e_(I_2)*e_(I_3) *x_I);
b2 = sub(basis(2,R),RS);  b6 = sub(basis(6,R),RS);  b8 = sub(basis(8,R),RS);
A1 = diff(diff((transpose b2),b6), myRules);
A2 = diff(transpose diff(diff((transpose b2),b6),b8), myRules);
A =sub( A2*A1,S);  f_2 = trace A;   time f_6 = trace A^3;
```

Evaluate these quickly at points.

Computing in the invariant ring $\mathbb{C}[\bigwedge^4\mathbb{C}^8]^{SL(8)} = \mathbb{C}[f_2, f_6, f_8, f_{10}, f_{12}, f_{14}, f_{18}]$

$\Delta_{\mathrm{Gr}(4,8)}$ has degree 126 in 70 variables. Might expect $\sim 10^{53}$ monomials.

However, there are only $15,976$ monomials spanning $\mathbb{C}[\bigwedge^4\mathbb{C}^8]^{\mathrm{SL}(8)}_{126}$.

It suffices to work on generic semi-simple element (only 7 variables).

Cluster computing; reduction mod 100 primes over 1000[1]; CRT, Rational Reconstruction produces

[1](each of these computations took approximately 4 hours, but we ran them in parallel).

Computing in the invariant ring $\mathbb{C}[\bigwedge^4\mathbb{C}^8]^{SL(8)} = \mathbb{C}[f_2, f_6, f_8, f_{10}, f_{12}, f_{14}, f_{18}]$

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Cluster computing; reduction mod 100 primes over 1000[1]; CRT, Rational Reconstruction produces

$$\begin{aligned}\Delta_{\mathrm{Gr}(4,8)} = &-(1122855063416382069258273636706506680023766222775944934559 8\\ &8613743812708107015862353 92/19003599762623464544744484198090 74\\ &880484088763429831167939681466204604687770731158447265625)f_2^{63}\\ &+\cdots+(3/16905146641687540708214291786189 09)f_{18}^{7},\end{aligned} \tag{1}$$

with a total of 15,942 terms.
Certified the result (standard lifting linear algebra mod several primes to rational linear algebra).

[1](each of these computations took approximately 4 hours, but we ran them in parallel).

Theorem (Holweck-Oeding 2021)

Consider a non-trivial vector space splitting $V = A \oplus B$. Let $X \subset \mathbb{P}V$ and $Y \subset \mathbb{P}A$ be projective varieties. Let π_B denote rational map $\mathbb{P}V \dashrightarrow \mathbb{P}A$ induced from the projection $V \to A$. If for each general smooth point[a] *$[y] \in Y$ there is a general smooth point $[x] \in X$ such that $\pi_B(\widehat{T}_x X) \subset \widehat{T}_y Y$ (the tangency condition), then*

$$Y^\vee \subseteq X^\vee \cap \mathbb{P}A^*.$$

Moreover if $X^\vee$ and $Y^\vee$ are hypersurfaces defined respectively by polynomials Δ_X and Δ_Y and, for every general point $[h] \in Y^\vee$, $H = \mathcal{V}(h)$, viewed as a hyperplane in $\mathbb{P}V$, is a point of multiplicity m of $X^\vee$ then

$$\Delta_Y^m \mid \operatorname{Res}(\Delta_X, A^*).$$

[a] By this term we mean points that are general among the smooth points, that is they are both smooth and avoid a closed proper subset that we do not always specify.

We can also slightly generalize a basic result of GKZ:

Corollary (Holweck-Oeding 2021)

The image of the rational map $Y_\pi := \pi_B(X)$ *satisfies the tangency condition, and hence* $Y_\pi^\vee \subset X^\vee \cap \mathbb{P}A^*$.

Corollary (Holweck-Oeding 2021)

Suppose $X \subset \mathbb{P}V$ is a variety and $V = A \oplus B$. Consider the following cases:

1. *Set $Y = \pi_B(X)$. Suppose $\widehat{X} \cap B = 0$ and $\dim \widehat{X} < dim B$.*
   (a) *Suppose the map $\pi_B \colon X \to \pi_B(X) = Y$ is an isomorphism of algebraic varieties.*
2. *Set $Y = X \cap \mathbb{P}A$. Suppose that for a general point $[y] \in Y$ there exists a smooth point $[x] \in X$ such that $\pi_B(\widehat{T}_x X) \subset \widehat{T}_y Y$.*

Then

$$Y^\vee \subseteq X^\vee \cap \mathbb{P}A^* \quad \text{in cases (1) and (2)} \quad \text{and} \quad Y^\vee = X^\vee \cap \mathbb{P}A^* \quad \text{in case (1a).}$$

If $X^\vee$ and $Y^\vee$ are hypersurfaces defined respectively by polynomials Δ_X and Δ_Y, then

$$\Delta_Z^m \mid \pi_B(\Delta_X) \quad \text{in cases (1) and (2)} \quad \text{and} \quad \Delta_Y^m \propto \pi_B(\Delta_X) \quad \text{in case (1a),}$$

where $m = mult_{[h]}\Delta_X$ for $[h]$ a general point of $Y^\vee$ viewed as a point in V^*. In particular, if there exist two distinct smooth points x_1, x_2 in X such that $\pi_B(\widehat{T}_{x_i} X) \subset \widehat{T}_y Y$, then $m \geq 2$.

Singular tensors: dual to unentangled states

[See Ottaviani's Intro to Hyperdeterminants]

- Let V_i be vector spaces (Hilbert spaces).
- A tensor $A \in V_0 \otimes \cdots \otimes V_p$ is *degenerate* if $\exists x^0 \otimes \cdots \otimes x^p$, with $x^i \in V_i$ such that

$$A\left(x^0, x^1, \ldots, V_i, \ldots, x^p\right) = 0 \qquad \forall i = 0, \ldots p \tag{$\star$}$$

  I.e. A describes a hyperplane in $V_0 \otimes \cdots \otimes V_p$ tangent to the Segre variety of decomposable tensors.
- Call $\mathcal{K}(A)$, the kernel, the set of non-zero $x^0 \otimes \cdots \otimes x^p$ satisfying ($\star$).
- Schläfli: Write $A(x) \in V_1 \otimes \cdots \otimes V_p$ as matrix depending linearly on x.
- If A is degenerate then $\det(A(x))$ is singular at v_0 for all $v_0 \otimes v_1 \otimes \cdots \otimes v_p \in \mathcal{K}(A)$.
- So $\mathrm{disc}\det(A(x)) = 0$, and this must divide the hyperdeterminant when A is degenerate.
- GKZ conjectured, and Weyman-Zelevinski proved, that $2 \times m \times m$, $3 \times m \times m$, $2 \times 2 \times 2 \times 2$ are the only formats where Schläfli gives exactly the hyperdeterminant. Otherwise get $Det(A) \cdot F$ for some high degree polynomial F.