G-Varieties and Principal Minors of Symmetric Matrices

Luke Oeding

Texas A&M University

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Dissertation Defense

Goals

- Let $G \subset GL(V)$, V- vector space over $\mathbb{C}$. A variety $X \subset \mathbb{P}V$ is a *G-variety* if $G.X \subset X$.
- Goal 1: Study a prototypical G-variety and learn how to study other G-varieties which arise in fields such as algebraic statistics, probability theory, signal processing, etc.).
- Goal 2: Solve the Holtz-Sturmfels Conjecture (set theoretic version).

Theorem

The variety of principal minors of symmetric matrices is cut out set theoretically by the hyperdeterminantal module.

Questions

- A principal minor of a matrix A is the determinant of a submatrix determined by striking out the same rows and columns of A, *i.e.* centered on the diagonal.
- Holtz and Schneider, D. Wagner: When is it possible to prescribe the principal minors of a symmetric matrix?
- Equivalently, when can you prescribe all the eigenvalues of a symmetric matrix and all of its principal submatrices?
- For $n \geq 3$ this is an overdetermined problem : $\binom{n+1}{2}$ versus 2^n.

The Variety of Principal Minors of Symmetric Matrices

- The variety of principal minors of $n \times n$ symmetric matrices, Z_n, is defined by the following rational map

$$\varphi : \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C}) \dashrightarrow \mathbb{P}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2) = \mathbb{P}\mathbb{C}^{2^n}$$

$$\begin{aligned}[A,t] \quad \mapsto \quad & \big[t^n, t^{n-1}\Delta_{[1,0\ldots,0]}(A), t^{n-1}\Delta_{[0,1,0,\ldots,0]}(A), t^{n-2}\Delta_{[1,1,0,\ldots,0]}(A), \\ & t^{n-1}\Delta_{[0,0,1,0,\ldots,0]}(A), t^{n-2}\Delta_{[1,0,1,0,\ldots,0]}(A), t^{n-2}\Delta_{[0,1,1,0,\ldots,0]}(A), \\ & t^{n-3}\Delta_{[1,1,1,0,\ldots,0]}(A), \cdots, \Delta_{[1,\ldots,1]}(A)\big]\end{aligned}$$

$\Delta_I(A) =$ principal minor with rows/columns indicated by the location of the 1's. Compact notation $\varphi([A,t]) = \left[t^{n-|I|}\Delta_I(A)X^I\right]$.

- Q: What are the Cartesian equations for Z_n?
- Expect many relations because of $\binom{n+1}{2}$ versus 2^n.

Examples

Compute principal minors

2 × 2 case: $$\left[\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{1,2} & a_{2,2} \end{pmatrix}, t\right] \mapsto [t^2, ta_{1,1}, ta_{2,2}, a_{1,1}a_{2,2} - a_{1,2}^2]$$

Work on set where $t = 1$, can go backwards:

$$[1, z_{10}, z_{01}, z_{11}] \mapsto \left[\begin{pmatrix} z_{10} & \pm\sqrt{z_{10}z_{01} - z_{11}^2} \\ \pm\sqrt{z_{10}z_{01} - z_{11}^2} & z_{01} \end{pmatrix}, 1\right]$$

- Can always solve the 2×2 case.
- In fact, in the $n \times n$ case, prescribing the 1×1 and 2×2 principal minors of a symmetric matrix *determines* the matrix up to signs of the off-diagonal terms.

Examples

$$\left[\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{1,2} & a_{2,2} & a_{2,3} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{pmatrix}, t\right]$$

$$\begin{aligned} \mapsto \quad & [t^3, t^2 a_{1,1}, t^2 a_{2,2}, t(a_{1,1}a_{2,2} - a_{1,2}^2), \\ & t^2 a_{3,3}, t(a_{1,1}a_{3,3} - a_{1,3}^2), t(a_{2,2}a_{3,3} - a_{1,2}^2), \\ & a_{1,1}a_{2,2}a_{3,3} + 2a_{1,2}a_{1,3}a_{2,3} - a_{1,1}a_{2,3}^2 - a_{2,2}a_{1,3}^2 - a_{3,3}a_{1,2}^2] \end{aligned}$$

Can you go backwards?

$$[z_{000}, z_{100}, z_{010}, z_{110}, z_{001}, z_{101}, z_{011}, z_{111}] \mapsto ??$$

Count parameters: $6 + 1$ matrix entries versus $7 + 1$ principal minors!
There must be at least one relation!

First result

Theorem (Holtz-Sturmfels '07)

$\mathcal{I}(Z_3) = \langle hyp \rangle$, *where hyp is...*

...this beautiful degree 4 homogeneous polynomial:

$$\begin{aligned} hyp = \quad & z_{000}^2 z_{111}^2 + z_{100}^2 z_{011}^2 + z_{010}^2 z_{101}^2 + z_{110}^2 z_{001}^2 \\ & + 4\big(z_{000} z_{110} z_{101} z_{011} + z_{100} z_{010} z_{001} z_{111}\big) \\ & - 2\big(z_{000} z_{100} z_{011} z_{111} + z_{000} z_{010} z_{101} z_{111} + z_{000} z_{001} z_{110} z_{111} \\ & + z_{100} z_{010} z_{011} z_{101} + z_{100} z_{001} z_{110} z_{011} + z_{001} z_{010} z_{101} z_{110}\big) \end{aligned}$$

-Cayley's hyperdeterminant of format $2 \times 2 \times 2$. Notice: *hyp* is a invariant under the action of $\mathfrak{S}_3 \times SL(2) \times SL(2) \times SL(2)$!

Hidden Symmetry

Theorem (Landsberg,Holtz-Sturmfels, -)

Z_n is invariant under the action of $G = \mathfrak{S}_n \times SL(2)^{\times n}$. Moreover, G is the largest subgroup of $GL(2^n)$ which preserves Z_n.

- *Fact:* A variety $X \subset \mathbb{P}^N$ is a G-variety $\Leftrightarrow$ the ideal $\mathcal{I}(X)$ is a G-module.
- Z_n is a subvariety of $\mathbb{P}(V_1 \otimes \cdots \otimes V_n)$, where each $V_i \simeq \mathbb{C}^2$.
- KEY POINT: We must study $\mathcal{I}(Z_n) \subset \mathrm{Sym}(V_1^* \otimes \cdots \otimes V_n^*)$ as a G-module!
- Mantra: "Each irreducible module is either in or out!"

Slight Detour: A Geometric Proof of Symmetry

- For non-degenerate $\omega \in \bigwedge^2 \mathbb{C}^n$, the Lagrangian Grassmannian is $Gr_\omega(n, 2n) = \{E \in Gr(n, 2n) \mid \omega(v, w) = 0 \ \forall v, w \in E\}$.
- $Gr_\omega(n, 2n)$ is a homogeneous variety for $Sp(2n)$.
- $Gr_\omega(n, 2n)$ is the image of the rational map:

$$\begin{array}{rcl} \psi : \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C}) & \dashrightarrow & \mathbb{P}\Gamma_n \simeq \mathbb{P}^{\binom{2n}{n} - \binom{2n}{n-2} - 1} \\ \{\textit{symmetric matrix}\} & \mapsto & \{\textit{vector of all nonredundant minors}\} \end{array}$$

- The connection: Z_n is a linear projection of $Gr_\omega(n, 2n)$.
- Can use this projection to *find* the symmetry group of Z_n as a subgroup of $Sp(2n)$.
- Try to find projections from homogeneous varieties to study other G-varieties.

Representation Theory

- Want to study $\mathcal{I}_d(Z_n) \subset S^d(V_1^* \otimes \cdots \otimes V_n^*)$.
- Each irreducible $\mathfrak{S}_n \times SL(2)^{\times n}$-module in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ is isomorphic to one indexed by partitions π_i of d of the form :

$$S_{\pi_1} S_{\pi_2} \dots S_{\pi_n} := \bigoplus_{\sigma \in \mathfrak{S}_n} S_{\pi_{\sigma(1)}} V_1^* \otimes S_{\pi_{\sigma(2)}} V_2^* \otimes \cdots \otimes S_{\pi_{\sigma(n)}} V_n^*$$

- If M is an irreducible G-module, then $M = \{G.v\}$, some vector v - use this as often as possible.
- This gives a finite list of vectors to test for ideal membership!
- Also gives a way to produce many polynomials in $\mathcal{I}(Z_n)$ from one polynomial.

Rephrasing of Previous Results

Theorem (Holtz-Sturmfels '07)

$\mathcal{I}(Z_3)$ is generated degree 4 by $\{hyp\} = S_{(2,2)}S_{(2,2)}S_{(2,2)}$.

Theorem (Holtz-Sturmfels '07)

$\mathcal{I}(Z_4)$ is generated in degree 4 by $\{\mathfrak{S}_4 \times SL(2)^{\times 4}.hyp_{123}\} = S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)}$.

hyp_{123} is the $2 \times 2 \times 2$ hyperdeterminant on the variables z_{***0}.

Conjecture (Holtz-Sturmfels '07)

$\mathcal{I}(Z_n)$ is generated in degree 4 by $\{\mathfrak{S}_n \times SL(2)^{\times n}.hyp_{123}\} = S_{(4)} \dots S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)}$.

A Limit of the Computer's Usefulness

- For $n = 3$: A single irreducible degree 4 polynomial on 8 variables cuts out the irreducible hypersurface in $\mathbb{P}^7$.
- For $n = 4$: 20 degree 4 polynomials on 16 variables. Macaulay2 $\Rightarrow$ the ideal is prime and has the correct dimension. But Z_4 is an irreducible variety + some facts from comm. alg. $\Rightarrow$ done.
- For $n = 5$: 250 degree 4 polynomials on 32 variables. Sadly, the computer has not yet told me whether or not this ideal is prime.
- For $n = 6$: 2500 degree 4 polynomials on 64 variables.
- For $n = n$: $\binom{n}{3}5^{n-3}$ degree 4 polynomials on 2^n variables. What can we say in general without the computer?

New Results

Theorem (-)

Let $M := \{\mathfrak{S}_n \times SL(2)^{\times n}.hyp_{123}\} = S_{(4)} \dots S_{(4)} S_{(2,2)} S_{(2,2)} S_{(2,2)}$*. The variety* Z_n *is cut out set theoretically by the hyperdeterminantal module.*

$$\mathcal{V}(M) = Z_n.$$

- To prove that $Z_n \subset \mathcal{V}(M)$, show that *hyp*, a highest weight vector for the irreducible module M, vanishes on every point of Z_n. Follows from 3×3 case.
- To prove that $Z_n \supset \mathcal{V}(M)$, a more geometric understanding of the zero set, $\mathcal{V}(M)$, is needed.

Outline of proof of main theorem

- Want to show $\mathcal{V}(M) \subset Z_n$ - do induction on n.
- Give a geometric characterization of $\mathcal{V}(M)$.
- Attempt to construct a matrix $A \in S^2\mathbb{C}^n$ that maps to $z \in \mathcal{V}(M)$.
- Identify possible obstructions as G-modules.
- Identify the space of obstructions geometrically.
- Show $\mathcal{V}(M)$ also contains the space of obstructions.

Characterizing the zero set of $\mathcal{V}(M)$ via augmentation

- Notice that $M_n = \underbrace{S_{(4)}\dots S_{(4)}}_{n-3} S_{(2,2)}S_{(2,2)}S_{(2,2)}$ and
  $M_{n+1} = \underbrace{S_{(4)}\dots S_{(4)}}_{n-2} S_{(2,2)}S_{(2,2)}S_{(2,2)}$ is still degree 4.
- What can we say about zero set of an augmented ideal $\mathcal{V}(\mathcal{I}_d(X)\otimes S^d V^*)$ based on $\mathcal{V}(\mathcal{I}_d(X))$?

Lemma (inspired by Landsberg-Manivel lemma regarding prolongation)

Let $X \subset \mathbb{P}W$ and let $\tilde{X} = \mathcal{V}(\mathcal{I}_d(X))$ (notation).

$$\mathcal{V}(\mathcal{I}_d(X)\otimes S^d V^*) = Seg(\tilde{X}\times \mathbb{P}V) \sqcup \bigcup_{L\subset \tilde{X}} \sigma_d(\mathbb{P}L\times \mathbb{P}V),$$

where $L\subset \tilde{X}$ are linear subspaces.

What does this buy us?

Consequence

Assume that $M = \bigoplus_i M_{\hat{i}} \otimes S^d V_i \subset S^d(V_1 \otimes \cdots \otimes V_n)$ *and* $V_i \simeq \mathbb{C}^2$*, then*

$$\mathcal{V}(M) = \cap_{i=1}^{n} \left(\bigcup_{L \subset V(M_{\hat{i}})} \mathbb{P}(L \otimes V_i) \right).$$

- Suppose $z \in \mathcal{V}(M) = \mathcal{V}(\bigoplus_i M_{\hat{i}} \otimes S^d V_i)$, and assume for induction that $\mathcal{V}(M_{\hat{i}}) \simeq Z_{n-1}$.
- Then our geometric realization gives n different expressions for z,

$$z = \varphi([A^{(i)}, t^{(i)}]) \otimes x_i^0 + \varphi([B^{(i)}, s^{(i)}]) \otimes x_i^1, \quad \text{(no summation)}$$

  where $A^{(i)}, B^{(i)}$ are $n-1 \times n-1$ matrices, and $\{x_i^0, x_i^1\} = V_i$.
- If we can use this information to build an $n \times n$ matrix A so that $\varphi([A, t]) = z$, we will have proved the theorem.

Building a matrix

We have n expressions

$$z = \varphi([A^{(i)}, t^{(i)}]) \otimes x_i^0 + \varphi([B^{(i)}, s^{(i)}]) \otimes x_i^1,$$

and the term $\varphi([A^{(1)}, t^{(1)}]) \otimes x_1^0$ can be thought of as the principal minors (not involving the first row and column) of the matrix

$$A(\overrightarrow{x_1}) = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ x_{1,2} & a_{1,2}^{(1)} & a_{2,2}^{(1)} & \dots & a_{2,n}^{(1)} \\ & \vdots & \vdots & \vdots & \\ x_{1,n} & a_{1,n}^{(1)} & a_{2,n}^{(1)} & \dots & a_{n,n}^{(1)} \end{pmatrix},$$

where $x_{1,i}$ are variables, and the entries of $A^{(1)} = (a_{i,j}^{(1)})$, are fixed.
The other expressions $\varphi([A^{(i)}, t^{(i)}]) \otimes x_i^0$ have a similar interpretation.

Building a matrix

- The 1×1 principal minors determine the diagonal entries and the 2×2 principal minors are all of the form $a_{i,i}a_{j,j} - a_{i,j}^2$ the 2×2 principal minors determine the off diagonal entries up to sign.
- We know that the principal minors $\Delta_I(A(\overrightarrow{x_i}))$ and $\Delta_I(A(\overrightarrow{x_j}))$ agree whenever $i, j \notin I$.
- Our question comes down to whether we can make consistent choices so that the matrices $A(\overrightarrow{x_i})$ agree.
- It suffices to prove that if we fix $A^{(1)}$, that we can choose $\overrightarrow{x_1}$ and $A^{(i)}$ so that all of the principal minors agree where the matrices overlap.

- Construct $A(\overrightarrow{x_i})^{(j)}$, by deleting the j^{th} row and column.
- By induction, it suffices to consider

$$A(x_{1,2}) = \begin{pmatrix} a_{1,1} & x_{1,2} & a_{1,3} & \dots & a_{1,n} \\ x_{1,2} & a_{2,2} & \dots & \dots & a_{2,n} \\ a_{1,3} & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \dots & & a_{n,n} \end{pmatrix},$$

  and show that we can pick $x_{1,2}$ so that all of the principal minors of $A(\overrightarrow{x_i})^{(j)}$ agree.
- We will have only determined that the matrix $A(x_{1,2})$ has all the correct principal minors (matching our point $z \in \mathcal{V}(M)$) except possibly the determinant.

Almost...

Lemma (The Almost Lemma, $n \geq 4.$)

Suppose $[z] = [z_I X^I] \in \mathcal{V}(M)$, *and* $[v_A] = [v_{A,I} X^I] = [\varphi([A,t])] \in Z_n$ *are such that* $z_I = v_{A,I}$ *for all* $I \neq [1,\dots,1]$. *If* $z_{[1,\dots,1]} \neq v_{A,[1,\dots,1]}$, *then*

$$[z] \in \bigcup_{\substack{|I_s| \leq 2 \\ 1 \leq s \leq m}} \left(Seg\left(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\right)\right) \subset Z_n.$$

We have essentially made a reduction to a problem in a single variable. Once the obstructions to solving this problem are identified as a G-module, the proof of this lemma is an application of the geometric characterization above.

Almost...but what does this buy me?

The lemma says that $Seg\left(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\right) \subset Z_n$.

- In fact, every point in $Seg\left(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\right) \subset Z_n$ comes from a block diagonal matrix with only 1×1 and 2×2 blocks.
- Such a matrix is a special case of a symmetric tri-diagonal matrix, and it's a fact that none of its principal minors depend on the sign of the off diagonal terms.
- We use this fact iteratively in our induction for the proof of the final lemma.

Concluding Remarks

- This problem shows how representation theory and geometry can be used to prove exciting new results.
- We resolved the set theoretic version of the Holtz-Sturmfels conjecture, but more work needs to be done in order to prove the ideal theoretic version.
- Thank you for attending! Special thanks to my thesis advisor, J.M. Landsberg and to the rest of my thesis committee.

Applications outside of geometry

- Spectral graph theory.
- Probability theory - covariance of random variables.
- Statistical physics - determinantal point processes.
- Matrix theory - P-matrices, GKK-τ matrices.