The Variety of Principal Minors of Symmetric Matrices and its Set Theoretic Defining Equations

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Goals

- Let $G \subset GL(V)$, V- vector space over $\mathbb{C}$. A variety $X \subset \mathbb{P}V$ is a *G-variety* if $G.X \subset X$.
- Goal 1: Study a prototypical G-variety and learn how to study other G-varieties which arise in fields such as algebraic statistics, probability theory, signal processing, etc.).
- Goal 2: Solve the Holtz-Sturmfels Conjecture (set theoretic version).

Questions

- A principal minor of a matrix A is the determinant of a submatrix formed by striking out the same rows and columns of A, *i.e.* centered on the diagonal.
- Holtz and Schneider, D. Wagner: When is it possible to prescribe the principal minors of a symmetric matrix?
- Equivalently, when can you prescribe all the eigenvalues of a symmetric matrix and all of its principal submatrices?
- For $n \geq 3$ this is an overdetermined problem : $\binom{n+1}{2}$ versus 2^n.

Examples: 2×2 case

Define a (homogeneous) map:
φ : symmetric matrices $\rightarrow$ principal minors:

$$\varphi\left(\begin{pmatrix} a & c \\ c & b \end{pmatrix}, t\right) = [t^2, ta, tb, ab - c^2]$$

When can we go backwards? Given $[w, x, y, z]$ is there a 2×2 matrix that has these principal minors? Need to solve: (WLOG assume $t = w = 1$)

$$\begin{aligned} x &= a \\ y &= b \\ z &= ab - c^2 \quad \Rightarrow \quad c = \pm\sqrt{xy - z} \end{aligned}$$

$$\varphi\left(\begin{pmatrix} x & \pm\sqrt{xy - z} \\ \pm\sqrt{xy - z} & y \end{pmatrix}, 1\right) = [1, x, y, z]$$

Conclude: Even in the $n \times n$ case, the 0×0, 1×1, and 2×2 minors *determine* a symmetric matrix up to the signs of the off-diagonal terms.

Examples 3×3:

$$\varphi\left(\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{pmatrix}, t\right)$$

$$\begin{array}{rl} & [t^3, t^2x_{11}, t^2x_{22}, t(x_{11}x_{22} - x_{12}^2), \\ = & t^2x_{33}, t(x_{11}x_{33} - x_{13}^2), t(x_{22}x_{33} - x_{23}^2), \\ & x_{11}x_{22}x_{33} + 2x_{12}x_{13}x_{23} - x_{11}x_{23}^2 - x_{22}x_{13}^2 - x_{33}x_{12}^2] \end{array}$$

Given $[X^{[000]}, X^{[100]}, X^{[010]}, X^{[110]}, X^{[001]}, X^{[101]}, X^{[011]}, X^{[111]}]$ is there a matrix that maps to it?

Count parameters: 7 versus 8 - there must be some relation that holds!

The First Relation

$$x_{12}^2 = X^{100}X^{010} - X^{110}$$
$$x_{13}^2 = X^{100}X^{001} - X^{101}$$
$$x_{23}^2 = X^{001}X^{010} - X^{011}$$

$$X^{111} = X^{100}X^{010}X^{001} - X^{100}x_{23}^2 - X^{010}x_{13}^2 - X^{001}x_{12}^2 + 2x_{12}x_{13}x_{23}$$

$$(X^{111} - X^{100}X^{010}X^{001} + X^{100}x_{23}^2 + X^{010}x_{13}^2 + X^{001}x_{12}^2)^2 = 4(x_{12}x_{13}x_{23})^2$$

$$\begin{pmatrix} X^{111} - X^{100}X^{010}X^{001} + X^{100}(X^{001}X^{010} - X^{011}) \\ +X^{010}(X^{100}X^{001} - X^{101}) + X^{001}(X^{100}X^{010} - X^{110}) \end{pmatrix}^2 = 4(X^{100}X^{010} - X^{110})(X^{100}X^{001} - X^{101})(X^{001}X^{010} - X^{011})$$

$$\begin{aligned} 0 = (X^{111})^2 + (X^{100})^2(X^{011})^2 + (X^{010})^2(X^{101})^2 + (X^{110})^2(X^{001})^2 \\ + 4X^{110}X^{101}X^{011} + 4X^{100}X^{010}X^{001}X^{111} \\ - 2X^{100}X^{011}X^{111} - 2X^{100}X^{010}X^{011}X^{101} - 2X^{010}X^{101}X^{111} \\ - 2X^{100}X^{001}X^{110}X^{011} - 2X^{001}X^{110}X^{111} - 2X^{001}X^{010}X^{101}X^{110} \end{aligned}$$

First result

Theorem (Holtz-Sturmfels '07)

All relations among the principal minors of a 3×3 matrix are generated by ...this beautiful degree 4 homogeneous polynomial:

$$
\begin{aligned}
&(X^{000})^2(X^{111})^2 + (X^{100})^2(X^{011})^2 \\
&+ (X^{010})^2(X^{101})^2 + (X^{110})^2(X^{001})^2 \\
&+ 4X^{000}X^{110}X^{101}X^{011} + 4X^{100}X^{010}X^{001}X^{111} \\
&- 2X^{000}X^{100}X^{011}X^{111} - 2X^{100}X^{010}X^{011}X^{101} \\
&- 2X^{000}X^{010}X^{101}X^{111} - 2X^{100}X^{001}X^{110}X^{011} \\
&- 2X^{000}X^{001}X^{110}X^{111} - 2X^{001}X^{010}X^{101}X^{110}
\end{aligned}
$$

-Cayley's hyperdeterminant of format $2 \times 2 \times 2$. Notice: It is invariant under the action of $\mathfrak{S}_3 \ltimes SL(2) \times SL(2) \times SL(2)$!

The Variety of Principal Minors of Symmetric Matrices

- The variety of principal minors of $n \times n$ symmetric matrices, Z_n, is defined by the following rational map

$$\varphi : \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C}) \dashrightarrow \mathbb{P}(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2) = \mathbb{P}\mathbb{C}^{2^n}$$

$$\begin{aligned}[A,t] \mapsto [t^n, t^{n-1}\Delta_{[10\ldots 0]}(A), t^{n-1}\Delta_{[010\ldots 0]}(A), t^{n-2}\Delta_{[110\ldots,0]}(A),\\ t^{n-1}\Delta_{[0010\ldots 0]}(A), t^{n-2}\Delta_{[1010\ldots 0]}(A), t^{n-2}\Delta_{[0110\ldots 0]}(A),\\ \ldots \quad \ldots, \Delta_{[1\ldots 1]}(A)]\end{aligned}$$

  where $\Delta_{[I]}(A)$ is the principal minor of A with rows indicated by I.
- Q: Given a vector v of length 2^n, how can you tell whether or not it arose in this way?
- A: test whether v satisfies all the relations in $\mathcal{I}(Z_n)$.

Hidden Symmetry

Theorem (Landsberg,Holtz-Sturmfels)

Z_n is invariant under the action of $G = \mathfrak{S}_n \ltimes SL(2)^{\times n}$.

- *Fact:* A variety $X \subset \mathbb{P}^N$ is a G-variety $\Leftrightarrow$ the ideal $\mathcal{I}(X)$ is a G-module.
- Z_n is a subvariety of $\mathbb{P}(V_1 \otimes \cdots \otimes V_n)$, where each $V_i \simeq \mathbb{C}^2$.
- KEY POINT: We must study $\mathcal{I}(Z_n) \subset \mathrm{Sym}(V_1^* \otimes \cdots \otimes V_n^*)$ as a G-module!
- Mantra: "Each irreducible module is either in or out!"

Slight Detour: A Geometric Proof of Symmetry

- For non-degenerate $\omega \in \bigwedge^2 \mathbb{C}^n$, the Lagrangian Grassmannian is $Gr_\omega(n, 2n) = \{E \in Gr(n, 2n) \mid \omega(v, w) = 0 \; \forall v, w \in E\}$.
- $Gr_\omega(n, 2n)$ is a homogeneous variety for $Sp(2n)$.
- $Gr_\omega(n, 2n)$ is the image of the rational map:

$$\begin{array}{rcl} \psi : \mathbb{P}(S^2\mathbb{C}^n \oplus \mathbb{C}) & \dashrightarrow & \mathbb{P}\Gamma_n \simeq \mathbb{P}^{\binom{2n}{n} - \binom{2n}{n-2} - 1} \\ \{symmetric\, matrix\} & \mapsto & \{vector\ of\ all\ nonredundant\ minors\} \end{array}$$

- The connection: Z_n is a linear projection of $Gr_\omega(n, 2n)$.
- Can use this projection to *find* the symmetry group of Z_n as a subgroup of $Sp(2n)$.
- Try to find projections of homogeneous varieties to study other G-varieties.

Multilinear Algebra

- $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ = homogeneous degree d polynomials on 2^n variables. It is a module for $G = SL(V_1) \times \cdots \times SL(V_n)$
- If we choose a basis $\{x_i^0, x_i^1\}$ of $V_i^* \simeq \mathbb{C}^2$ for each i, then $V_1^* \otimes \cdots \otimes V_n^*$ has the induced basis $x_1^{\epsilon_1} \otimes \cdots \otimes x_n^{\epsilon_n} =: X^I$.
- Then G acts on $V_1^* \otimes \cdots \otimes V_n^*$ by change of basis in each factor: If $g = (g_1, \ldots, g_n) \in G$, then

$$g.X^I = (g_1.x_1^{\epsilon_1}) \otimes \cdots \otimes (g_n.x_n^{\epsilon_n}),$$

and acts on $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ by the induced action:

$$g.(X^I X^J \ldots X^K) = (g.X^I)(g.X^J) \ldots (g.X^K)$$

- We have defined the action on a basis of each module, so we can just extend by linearity to get the action on the whole module.

Representation Theory

- Want to study $\mathcal{I}_d(Z_n) \subset S^d(V_1^* \otimes \cdots \otimes V_n^*)$.
- Each irreducible $\mathfrak{S}_n \ltimes SL(2)^{\times n}$-module in $S^d(V_1^* \otimes \cdots \otimes V_n^*)$ is isomorphic to one indexed by partitions π_i of d of the form :

$$S_{\pi_1} S_{\pi_2} \dots S_{\pi_n} := \bigoplus_{\sigma \in \mathfrak{S}_n} S_{\pi_{\sigma(1)}} V_1^* \otimes S_{\pi_{\sigma(2)}} V_2^* \otimes \cdots \otimes S_{\pi_{\sigma(n)}} V_n^*$$

- Can use the combinatorial information $\pi_1, \dots, \pi_n$ to construct the module.
- If M is an irreducible G-module, then $M = \{G.v\}$, some vector v - use this as often as possible.
- This gives a finite list of vectors to test for ideal membership!
- Also gives a way to produce many polynomials in $\mathcal{I}(Z_n)$ from one polynomial.

An Example

The module $S_{(2,2)}V \subset V^{\otimes 4}$ is one dimensional, and every vector is a scalar multiple of

$$h = 2X^{0011} - X^{1001} - X^{1010} - X^{0101} - X^{0110} + 2X^{1100}$$

To find a polynomial in $S_{(2,2)}V_1 \otimes S_{(2,2)}V_2 \otimes S_{(2,2)}V_3$, we need to compute $h \otimes h \otimes h$ in $V_1^{\otimes 4} \otimes V_2^{\otimes 4} \otimes V_3^{\otimes 4}$, but we want a polynomial in $S^4(V_1 \otimes V_2 \otimes V_3)$, so we just permute

$$V_1^{\otimes 4} \otimes V_2^{\otimes 4} \otimes V_3^{\otimes 4} \to (V_1 \otimes V_2 \otimes V_3)^{\otimes 4}$$

and symmetrize

$$(V_1 \otimes V_2 \otimes V_3)^{\otimes 4} \to S^4(V_1 \otimes V_2 \otimes V_3)$$

An Example

Finally, we get the result

$$
\begin{aligned}
(X^{000})^2(X^{111})^2 + (X^{100})^2(X^{011})^2 \\
+ (X^{010})^2(X^{101})^2 + (X^{110})^2(X^{001})^2 \\
+ 4X^{000}X^{110}X^{101}X^{011} + 4X^{100}X^{010}X^{001}X^{111} \\
- 2X^{000}X^{100}X^{011}X^{111} - 2X^{100}X^{010}X^{011}X^{101} \\
- 2X^{000}X^{010}X^{101}X^{111} - 2X^{100}X^{001}X^{110}X^{011} \\
- 2X^{000}X^{001}X^{110}X^{111} - 2X^{001}X^{010}X^{101}X^{110}
\end{aligned}
$$

In fact, this is Cayley's hyperdeterminant of format $2 \times 2 \times 2$!
It's an irreducible degree 4 polynomial on 8 variables.
It is invariant under the action of $\mathfrak{S}_3 \ltimes SL(2) \times SL(2) \times SL(2)$.
It generates the module $S_{(2,2)}S_{(2,2)}S_{(2,2)}$.
It is the single equation defining the hypersurface Z_3.

Rephrasing of Previous Results

Theorem (Holtz-Sturmfels)

$\mathcal{I}(Z_3)$ is generated in degree 4 by $S_{(2,2)}S_{(2,2)}S_{(2,2)}$ (Cayley's Hyperdeterminant of format $2 \times 2 \times 2$).

Theorem (H-S)

$\mathcal{I}(Z_4)$ is generated in degree 4 by $S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)}$ (A hyperdeterminantal module).

Remark: $S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)}$ is the span of the G-orbit of the $2 \times 2 \times 2$ hyperdeterminant on the variables $X^{[***0]}$.

Conjecture (H-S)

$\mathcal{I}(Z_n)$ is generated in degree 4 by $S_{(4)} \dots S_{(4)}S_{(2,2)}S_{(2,2)}S_{(2,2)}$ (the hyperdeterminantal module).

A Limit of the Computer's Usefulness

- For $n = 3$: A single irreducible degree 4 polynomial on 8 variables cuts out the irreducible hypersurface in $\mathbb{P}^7$.
- For $n = 4$: 20 degree 4 polynomials on 16 variables. Macaulay2 $\Rightarrow$ the ideal is prime and has the correct dimension. But Z_4 is an irreducible variety + some facts from comm. alg. $\Rightarrow$ done.
- For $n = 5$: 250 degree 4 polynomials on 32 variables. Sadly, the computer has not yet told me whether or not this ideal is prime.
- For $n = 6$: 2500 degree 4 polynomials on 64 variables. :-(
- For $n = n$: $\binom{n}{3}5^{n-3}$ degree 4 polynomials on 2^n variables. What can we say in general without the computer?

New Results

Theorem (-)

Let $HD := \{\mathfrak{S}_n \ltimes SL(2)^{\times n}.hyp_{123}\} = S_{(4)} \ldots S_{(4)} S_{(2,2)} S_{(2,2)} S_{(2,2)}$. *The variety* Z_n *is cut out set theoretically by the hyperdeterminantal module.*

$$\mathcal{V}(HD) = Z_n.$$

- To prove that $Z_n \subset \mathcal{V}(HD)$, show that *hyp*, a highest weight vector for the irreducible module M, vanishes on every point of Z_n. Follows from 3×3 case.
- To prove that $Z_n \supset \mathcal{V}(HD)$, a more geometric understanding of the zero set, $\mathcal{V}(HD)$, is needed.

Outline of proof of main theorem

- Want to show $\mathcal{V}(HD) \subset Z_n$ - do induction on n.
- Give a geometric characterization of $\mathcal{V}(HD)$.
- Attempt to construct a matrix $A \in S^2\mathbb{C}^n$ that maps to $z \in \mathcal{V}(HD)$.
- Identify possible obstructions as G-modules.
- Identify the space of obstructions geometrically.
- Show $\mathcal{V}(HD)$ also contains the space of obstructions.

Applications outside of geometry

- Spectral graph theory.
- Probability theory - covariance of random variables.
- Statistical physics - determinantal point processes.
- Matrix theory - P-matrices, GKK-τ matrices.

Spectral Graph Theory

Let Γ be a graph with

- vertex set $Q_0 = \{v_1, \dots, v_n\}$
- edge set $Q_1 = \{e_{i,j} \mid \overrightarrow{v_i v_j} \in \Gamma\}$.

The graph Laplacian of an undirected graph is a matrix

$$\Delta(\Gamma)_{i,j} = \begin{cases} -1 & \text{if } i \neq j \text{ and } e_{i,j} \in Q_1 \\ 0 & \text{if } i \neq j \text{ and } e_{i,j} \notin Q_1 \\ deg(v_i) & \text{if } i = j \end{cases}$$

The principal minors of $\Delta(\Gamma)$ are invariants of the graph, in fact:

Theorem (Kirchoff's Matrix-Tree theorem ($\sim$1850's))

Any $(n-1) \times (n-1)$ principal minor of $\Delta(\Gamma)$ counts the number of spanning trees of Γ.

Spectral Graph Theory

There are many generalizations of the Matrix-Tree Theorem, such as

Theorem (Matrix-Forest Theorem)

$\Delta(\Gamma)_S^S =$ number of spanning forests of Γ rooted at vertices indexed by S, where $\Delta(\Gamma)_S^S$ is the principal minor of $\Delta(\Gamma)$ indexed by S.

The $\Delta(\Gamma)_S^S$ are graph invariants. The relations among principal minors are then also relations among these graph invariants.

Corollary (Corollary to Main Theorem)

There exists an undirected weighted graph Γ with invariants $[v] \in \mathbb{P}^{2^n-1}$ specified by the principal minors of a symmetric matrix $\Delta_{wt}(\Gamma)$ if and only if $[v]$ is a zero of all the polynomials in the hyperdeterminantal module.

Concluding Remarks

- This problem shows how representation theory and geometry can be used to prove exciting new results.
- We resolved the set theoretic version of the Holtz-Sturmfels conjecture, but more work needs to be done in order to prove the ideal theoretic version.
- Thank you for attending! Special thanks to my thesis advisor, J.M. Landsberg.

Characterizing the zero set of $\mathcal{V}(HD)$ via augmentation

- Notice that $HD_n = \underbrace{S_{(4)} \dots S_{(4)}}_{n-3} S_{(2,2)} S_{(2,2)} S_{(2,2)}$ and $HD_{n+1} = \underbrace{S_{(4)} \dots S_{(4)}}_{n-2} S_{(2,2)} S_{(2,2)} S_{(2,2)}$ is still degree 4.
- What can we say about zero set of an augmented ideal $\mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*)$ based on $\mathcal{V}(\mathcal{I}_d(X))$?

Lemma (inspired by Landsberg-Manivel lemma regarding prolongation)

Let $X \subset \mathbb{P}W$ and let $\tilde{X} = \mathcal{V}(\mathcal{I}_d(X))$ (notation).

$$\mathcal{V}(\mathcal{I}_d(X) \otimes S^d V^*) = Seg(\tilde{X} \times \mathbb{P}V) \cup \bigcup_{L \subset \tilde{X}} \mathbb{P}(L \otimes V),$$

where $L \subset \tilde{X}$ are linear subspaces.

What does this buy us?

Consequence

Assume that $HD = \bigoplus_i HD_{\hat{i}} \otimes S^d V_i \subset S^d(V_1 \otimes \cdots \otimes V_n)$ *and* $V_i \simeq \mathbb{C}^2$*, then*

$$\mathcal{V}(HD) = \cap_{i=1}^{n} \left(\bigcup_{L \subset V(HD_{\hat{i}})} \mathbb{P}(L \otimes V_i) \right).$$

- Suppose $z \in \mathcal{V}(HD) = \mathcal{V}(\bigoplus_i HD_{\hat{i}} \otimes S^d V_i)$, and assume for induction that $\mathcal{V}(HD_{\hat{i}}) \simeq Z_{n-1}$.
- Then our geometric realization gives n different expressions for z,

$$z = \varphi([A^{(i)}, t^{(i)}]) \otimes x_i^0 + \varphi([B^{(i)}, s^{(i)}]) \otimes x_i^1, \quad \text{(no summation)}$$

  where $A^{(i)}, B^{(i)}$ are $n-1 \times n-1$ symmetric matrices, and $\{x_i^0, x_i^1\} = V_i$.
- If we can use this information to build an $n \times n$ matrix A so that $\varphi([A, t]) = z$, we will have proved the theorem.

Building a matrix

We have n expressions

$$z = \varphi([A^{(i)}, t^{(i)}]) \otimes x_i^0 + \varphi([B^{(i)}, s^{(i)}]) \otimes x_i^1,$$

and the term $\varphi([A^{(1)}, t^{(1)}]) \otimes x_1^0$ can be thought of as the principal minors (not involving the first row and column) of the matrix

$$A(\overrightarrow{x_1}) = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ x_{1,2} & a_{1,2}^{(1)} & a_{2,2}^{(1)} & \dots & a_{2,n}^{(1)} \\ & \vdots & \vdots & \vdots & \\ x_{1,n} & a_{1,n}^{(1)} & a_{2,n}^{(1)} & \dots & a_{n,n}^{(1)} \end{pmatrix},$$

where $x_{1,i}$ are variables, and the entries of $A^{(1)} = (a_{i,j}^{(1)})$, are fixed. The other expressions $\varphi([A^{(i)}, t^{(i)}]) \otimes x_i^0$ have a similar interpretation.

Building a matrix

- The 1×1 principal minors determine the diagonal entries and the 2×2 principal minors are all of the form $a_{i,i}a_{j,j} - a_{i,j}^2$ the 2×2 principal minors determine the off diagonal entries up to sign.
- We know that the principal minors $\Delta_I(A(\overrightarrow{x_i}))$ and $\Delta_I(A(\overrightarrow{x_j}))$ agree whenever $i, j \notin I$.
- Our question comes down to whether we can make consistent choices so that the matrices $A(\overrightarrow{x_i})$ agree.
- It suffices to prove that if we fix $A^{(1)}$, that we can choose $\overrightarrow{x_1}$ and $A^{(i)}$ so that all of the principal minors agree where the matrices overlap.

- Construct $A(\overrightarrow{x_i})^{(j)}$, by deleting the j^{th} row and column.
- By induction, it suffices to consider

$$A(x_{1,2}) = \begin{pmatrix} a_{1,1} & x_{1,2} & a_{1,3} & \dots & a_{1,n} \\ x_{1,2} & a_{2,2} & \dots & \dots & a_{2,n} \\ a_{1,3} & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \dots & & a_{n,n} \end{pmatrix},$$

  and show that we can pick $x_{1,2}$ so that all of the principal minors of $A(\overrightarrow{x_i})^{(j)}$ agree.
- We will have only determined that the matrix $A(x_{1,2})$ has all the correct principal minors (matching our point $z \in \mathcal{V}(HD)$) except possibly the determinant.

Almost...

Lemma (The Almost Lemma, $n \geq 4$.)

Suppose $[z] = [z_I X^I] \in \mathcal{V}(HD)$, *and* $[v_A] = [v_{A,I} X^I] = [\varphi([A,t])] \in Z_n$ *are such that* $z_I = v_{A,I}$ *for all* $I \neq [1,\ldots,1]$. *If* $z_{[1,\ldots,1]} \neq v_{A,[1,\ldots,1]}$, *then*

$$[z] \in \bigcup_{\substack{|I_s| \leq 2 \\ 1 \leq s \leq m}} \left(Seg\left(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\right)\right) \subset Z_n.$$

We have essentially made a reduction to a problem in a single variable. Once the obstructions to solving this problem are identified as a G-module, the proof of this lemma is an application of the geometric characterization above.

Almost...but what does this buy me?

The lemma says that $Seg\big(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\big) \subset Z_n$.

- In fact, every point in $Seg\big(\mathbb{P}V_{I_1} \times \cdots \times \mathbb{P}V_{I_m}\big) \subset Z_n$ comes from a block diagonal matrix with only 1×1 and 2×2 blocks.
- Such a matrix is a special case of a symmetric tri-diagonal matrix, and it's a fact that none of its principal minors depend on the sign of the off diagonal terms.
- We use this fact iteratively in our induction for the proof of the final lemma.