Eigenvectors of Tensors and Waring Decomposition

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Polynomial Waring decomposition

Let $V \cong \mathbb{C}^{n+1}$, $f \in S^d V$ – homogeneous polynomial.

Waring decomposition: $f = \sum_{i=1}^r c_i v_i^d$, with $c_i \in \mathbb{C}$, and $v_i \in V$.

Goals:

- Algorithms that quickly decompose low rank forms.
  (naive algorithms always exist, but are infeasible)
- Uniform treatment (Eigenvectors and vector bundles).

Non-Goal:

- One algorithm to decompose them all (NP-hard! -[Lim-Hillar'12]).

Motivation:

- CDMA-like communication scheme:
  Send (the coefficients of) $f = \sum_{i=1}^r c_i v_i^d$.
  Recover v_i uniquely.

Main Results

Theorem (O.-Ottaviani '13)

Let $f \in S^d\mathbb{C}^{n+1}$, with $d = 2m+1$, $n+1 \geq 4$, and general among forms of rank $\leq r$. If $r \leq \binom{m+n}{n}$ then the Koszul Flattening Algorithm produces the unique Waring decomposition.

We implemented our algorithm in Macaulay2 and you can download it from the ancillary files accompanying the arXiv version of our paper.

Algebraic Geometry helps Engineering: generic rank ($/\mathbb{C}$)

Theorem (Campbell 1891, Terracini 1916, Alexander-Hirschowitz 1995)

The general $f \ inS^d\mathbb{C}^{n+1}$, $d \geq 3$ has rank

$$\left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil, \qquad \text{the generic rank, except}$$

- $2 \leq n \leq 4, d = 4$ – *generic rank is* $\binom{n+2}{2}$,
- $(n, d) = (4, 3)$ – *generic rank is 8.*

Algebraic Geometry helps Engineering: Uniqueness ($/\mathbb{C}$)

Theorem (... A-H, '95)

The general f in$S^d\mathbb{C}^{n+1}$, $d \geq 3$ has the generic rank $\left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil$, except

- $2 \leq n \leq 4, d = 4$ – *generic rank is* $\binom{n+2}{2}$,
- $(n, d) = (4, 3)$ – *generic rank is 8.*

Theorem (Sylvester 1851, Chiantini-Ciliberto, Mella, Ballico 2002-2005)

The general $f \in S^d\mathbb{C}^{n+1}$ among the forms of subgeneric rank has a unique decomposition, except

- $2 \leq n \leq 4$, $d = 4$, $r = \binom{n+2}{2} - 1$, ∞*-ly many decomps.* defective
- $(n, d) = (4, 3)$, $r = 7$, ∞*-ly many decomps.* defective
- *rank* 9 *in* $S^6\mathbb{C}^3$, 2 *decomps.* weakly defective
- *rank* 8 *in* $S^4\mathbb{C}^4$, 2 *decomps.* weakly defective

Algebraic Geometry helps Engineering: Non-Uniqueness ($/\mathbb{C}$)

Expected: If $\frac{\binom{n+d}{d}}{n+1}$ is an integer, then uniqueness fails for the general form.

Mella showed in 2006 that when $d > n$ this is true.

The only known failures are (and we give a uniform proof):

- $S^{2m+1}\mathbb{C}^2$ rank $m+1$ Sylvester 1851,
- $S^5\mathbb{C}^3$ rank 7 Hilbert-Palatini-Richmond 1902,
- $S^3\mathbb{C}^4$ rank 5 Sylvester Pentahedral Theorem.

From equations to decompositions

General approach:

- Find nice (determinantal) equations for secant varieties
- Get an algorithm for decomposition.

A Flattenings / Catalecticants / truncated moment matrices

B Koszul Flattenings and eigenvectors of tensors

Koszul Flattenings: Examples / Overview

Equations of secant varieties from Koszul flattenings:

- Strassen: $\sigma_r(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}\left(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3\right)$
- Toeplitz: $\sigma_r(\mathbb{P}^2 \times \nu_2(\mathbb{P}^3)) \subset \mathbb{P}\left(\mathbb{C}^3 \otimes S^2\mathbb{C}^4\right)$
- Aronhold: $\sigma_r(\nu_3(\mathbb{P}^2)) \subset \mathbb{P}\left(S^3\mathbb{C}^3\right)$
- Cartwright-Erman-O.'11:
  $$\sigma_r(\mathbb{P}^2 \times \nu_2(\mathbb{P}^n)) \subset \mathbb{P}\left(\mathbb{C}^3 \otimes S^2\mathbb{C}^{n+1}\right), r \leq 5.$$
- Landsberg-Ottaviani 2012: Many more cases, much more general.

Our decomposition algorithms via Koszul Flattenings

- Sylvester Pentahedral Thm.: $S^3\mathbb{C}^4$, $r \leq 5$,
- HPR quintics: $S^5\mathbb{C}^3$, $r \leq 7$,
- More generally: $S^{2m+1}\mathbb{C}^{n+1}$, $r \leq \binom{n+m}{n}$.

Review: The catalecticant algorithm via an example

Decompose $f = 7x^3 - 30x^2y + 42xy^2 - 19y^3 \in S^3(\mathbb{C}^2)$:
Compute the flattening:

$$S^2(\mathbb{C}^2)^* \quad \xrightarrow{C_f} \quad \mathbb{C}^2,$$

$C_f = \begin{pmatrix} 7 & -10 & 14 \\ -10 & 14 & -19 \end{pmatrix}$, with kernel: $\left\{ \begin{pmatrix} 6 \\ 7 \\ 2 \end{pmatrix} \right\}$.

The kernel K (in the space of polynomials on the dual) is spanned by

$$6\partial_x^2 + 7\partial_x\partial_y + 2\partial_y^2 = (2\partial_x + \partial_y)(3\partial_x + 2\partial_y).$$

Notice $(2\partial_x + \partial_y)$ kills $(-x + 2y)$ and $(-x + 2y)^d$ for all d.
Also, $(3\partial_x + 2\partial_y)$ kills $(2x - 3y)$ and $(2x - 3y)^d$ for all d.
K annihilates precisely (up to scalar) $\{(-x + 2y), (2x - 3y)\}$.

Therefore $f = c_1(-x + 2y)^3 + c_2(2x - 3y)^3$.

Solve: $c_1 = c_2 = 1$.

Catalecticant algorithm in general [Iarrobino-Kanev 1999]

Input: $f \in S^d(V) \quad V = \mathbb{C}^{n+1}$.

1. Construct $C_f^m = C_f, \quad m = \lceil \frac{d}{2} \rceil$

$$\begin{array}{rcl} C_f^m\colon S^m V^* & \longrightarrow & S^{d-m} V \\ x_{i_1} \cdots x_{i_m} & \longmapsto & \dfrac{\partial^m f}{\partial_{x_{i_1}} \cdots \partial_{x_{i_m}}} \end{array}$$

2. Compute $\ker C_f$, note $\mathrm{Rank}(f) \geq \mathrm{rank}(C_f)$.
3. Compute $Z' = zeros(\ker C_f)$
   – if $\#Z' = \infty$, fail
   – else $Z' = \{[v_1], \ldots, [v_s]\}$
4. Solve

$$f = \sum_{i=1}^{s} c_i v_i^d, \qquad c_i \in \mathbb{C}.$$

Output: The unique Waring decomposition of f.

Catalecticant algorithm in general [Iarrobino-Kanev 1999]

The catalecticant algorithm appears in work of Sylvester, Iarrobino-Kanev, Brachat-Comon-Mourrain-Tsigaridas, Bernardi-Idá-Gimigliano.
Iarrobino and Kanev gave bounds for the success of the catalecticant algorithm. Here is a slight improvement:

Theorem (O.-Ottaviani 2013)

Let $\sum_{i=1}^{r} v_i^d = f$ be general among forms of rank r in $S^d V$. Set $z_i := [v_i]$, $Z := \{z_1, \ldots, z_r\}$ and let $m = \lceil \frac{d}{2} \rceil$.

1. *If d is even and $r \leq \binom{n+m}{n} - n - 1$,*
   or if d is odd and $\leq \binom{n+m-1}{n}$,
   then $\ker C_f = I_{Z,m}$ (subspace of deg. m polys vanishing on Z).
   $\Rightarrow$ the catalecticant algorithm succeeds with $Z = Z' = zeros(\ker C_f)$.
2. *If d is even $n \geq 3$ and $r = \binom{n+m}{n} - n$, $Z \subsetneq Z'$ is possible.*
   $\Rightarrow$ the catalecticant algorithm succeeds after finitely many checks.

Why the catalecticant algorithm works

Given $f \in S^d V$, we have the catalecticant:

$$C_f^m \colon S^m V^* \longrightarrow S^{d-m} V$$
$$x_{i_1} \cdots x_{i_m} \longmapsto \frac{\partial^m f}{\partial_{x_{i_1}} \cdots \partial_{x_{i_m}}}$$

Rank conditions:

f has rank 1 $\Rightarrow$ rank $C_f = 1$.
subadditivity of matrix rank implies that
(f has rank $r \Rightarrow$ rank $C_f \leq r$).

The zero set of the kernel is polar to the linear forms in the decomposition:

Notice that $\frac{\partial}{\partial(\alpha x + \beta y)} \cdot (\beta x - \alpha y)^d = 0$ ($\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y}$ is apolar to $\beta x - \alpha y$).

In the case of binary forms, a general elt. F of the kernel factors (FTA).
i.e. $F = l_1^{\perp} \cdots l_r^{\perp}$ kills all linear forms in decomposition.

There exist $c_i \in \mathbb{C}$ such that $f = \sum_{i=1}^{r} c_i l_i^d$ if and only if $l_1^{\perp} \cdots l_r^{\perp} f = 0$.
One inclusion is obvious, the other is by dimension count.

Eigenvectors of tensors

An essential ingredient is the notion of an eigenvector of a tensor.

The eigenvector equation for matrices: $M \in \mathbb{C}^{n\times n}, v \in \mathbb{C}^n$,

$$Mv = \lambda v, \quad \lambda \in \mathbb{C} \qquad \Longleftrightarrow \qquad M(v) \wedge v = 0$$

Definition

Let $M \in \mathrm{Hom}(S^m V, \bigwedge^a V)$. $v \in V$ is an eigenvector of the tensor M if

$$M(v^m) \wedge v = 0.$$

When $a = m = 1$ this is the classical definition.
When $a = 1$, [Lim'05] and [Qi'05] independently introduced this notion.
Further generalizations: Ottaviani-Sturmfels, Sam (Kalman varieties), and Qi et.al. (Spectral theory of tensors).

The number of eigenvectors of different types of tensors

Theorem (O.-Ottaviani '13)

For a general $M \in \mathrm{Hom}(S^m\mathbb{C}^{n+1}, \bigwedge^a \mathbb{C}^{n+1})$ *the number* $e(M)$ *of eigenvectors is*

$$\boxed{\begin{array}{c} e(M) = m, \textit{ when } n = 1 \textit{ and } a \in \{0, 2\}, \\ e(M) = \infty, \textit{ when } n > 1 \textit{ and } a \in \{0, n+1\}, \\ \textit{(classical)} \end{array}}$$

$$\boxed{\begin{array}{ll} e(M) = \frac{m^{n+1}-1}{m-1}, & \textit{when } a = 1 \textit{ [CS'10]}, \\ e(M) = 0, & \textit{for } 2 \le a \le n-2, \\ e(M) = \frac{(m+1)^{n+1}+(-1)^n}{m+2}, & \textit{for } a = n-1. \end{array}}$$

Our result includes a result of Cartwright-Sturmfels. Our proofs rely on the simple observation that the a Chern class computation for the appropriate vector bundle gives the number of eigenvectors.

The Koszul complex and Koszul matrices

The Koszul complex arises via the minimal free resolution of the maximal ideal $\langle x_0, \ldots, x_n \rangle$. Let V be the span of the x_i.

$$0 \longrightarrow \bigwedge^{n+1} V \xrightarrow{k_{n+1}} \bigwedge^{n} V \longrightarrow \cdots \xrightarrow{k_3} \bigwedge^{2} V \xrightarrow{k_2} \bigwedge^{1} V \xrightarrow{k_1} \mathbb{C} \longrightarrow 0$$

Some examples:

for $n = 2$, $k_1 = \begin{pmatrix} w & x & y \end{pmatrix}$, $k_2 = \begin{pmatrix} -x & -y & 0 \\ w & 0 & -y \\ 0 & w & x \end{pmatrix}$ $k_3 = \begin{pmatrix} y \\ -x \\ w \end{pmatrix}$,

for $n = 3$, $k_1 = \begin{pmatrix} w & x & y & z \end{pmatrix}$, $k_2 = \begin{pmatrix} -x & -y & 0 & -z & 0 & 0 \\ w & 0 & -y & 0 & -z & 0 \\ 0 & w & x & 0 & 0 & -z \\ 0 & 0 & 0 & w & x & y \end{pmatrix}$, ...

Note: these complexes are exact.

Sections of vector bundles to eigenvectors of tensors

Construct a map (tensor a Koszul map with a catalecticant map)

$$A_f \colon \operatorname{Hom}(S^m V, \textstyle\bigwedge^a V) \longmapsto \operatorname{Hom}(\textstyle\bigwedge^{n-a} V, S^{d-m-1} V)$$

$M \in \operatorname{Hom}(S^m V, \bigwedge^a V)$, v is an eigenvector of M iff $M(v^m) \wedge v = 0$.

Lemma

$M \in \operatorname{Hom}(S^m V, \bigwedge^a V)$,

1. *v is an eigenvector of M iff $M \in \ker A_f$.*
2. *Let $f = \sum_{i=1}^r v_i^d$. If each v_i is an eigenvector of M, then $M \in \ker A_f$.*

Lemma

Let Q be the quotient bundle on $\mathbb{P}^n$.

1. *The fiber of $\bigwedge^a Q$ at $x = [v]$ is isomorphic to* $\operatorname{Hom}([v^m], \bigwedge^a V / \langle v \wedge \bigwedge^{a-1} V \rangle$.
2. *the section s_M vanishes if and only if v is an eigenvector of M.*

Koszul Algorithm examples: HPR Quinitics

Let $V = \mathbb{C}^3$ – a general form $f \in S^5\mathbb{C}^3$ has rank 7.
Catalecticants:

$$C_f\colon S^3V^* \longrightarrow S^2V$$

is a 6×10 matrix - with max rank 6, so too small to detect rank 7.
Koszul Flattening:
$S^5V \subset S^2V \otimes V \otimes S^2V \leftarrow S^2V \otimes \bigwedge^2 V \otimes V^* \otimes S^2V$.
Get a map:

$$\begin{array}{rcl} A_f\colon S^2V^* \otimes \bigwedge^2 V^* & \longrightarrow & V^* \otimes S^2V \\ \mathrm{Hom}(S^2V, V) & \longrightarrow & \mathrm{Hom}(V, S^2V) \end{array}$$

$$A_f = \begin{pmatrix} -x & -y & 0 \\ w & 0 & -y \\ 0 & w & x \end{pmatrix} \otimes C_f = \begin{pmatrix} -C_{f_x} & -C_{f_y} & 0 \\ C_{f_w} & 0 & -C_{f_y} \\ 0 & C_{f_w} & C_{f_x} \end{pmatrix},$$

where C_{f_z} is the 6×6 catalecticant of $\frac{\partial f}{\partial z}$.

Koszul Algorithm examples: HPR Quinitics

Koszul Flattening:

$S^5V \subset S^2V \otimes V \otimes S^2V \leftarrow S^2V \otimes \bigwedge^2 V \otimes V^* \otimes S^2V$.

Get a map:

$$\begin{array}{rcl} A_f : S^2V^* \otimes \bigwedge^2 V^* & \longrightarrow & V^* \otimes S^2V \\ \mathrm{Hom}(S^2V, V) & \longrightarrow & \mathrm{Hom}(V, S^2V) \end{array}$$

$$A_f = \begin{pmatrix} -x & -y & 0 \\ w & 0 & -y \\ 0 & w & x \end{pmatrix} \otimes C_f = \begin{pmatrix} -C_{f_x} & -C_{f_y} & 0 \\ C_{f_w} & 0 & -C_{f_y} \\ 0 & C_{f_w} & C_{f_x} \end{pmatrix},$$

- A_f is skew-symmetrizable, so even has rank.
- If f has rank 7, A_f has rank ≤ 14.
- The 16×16 Pfaffians vanish on the locus of border rank 7 forms.
- The general M in $\mathrm{Hom}(S^2V, V)$ has 7 eigenvectors, [Cartwright-Sturmfels].
- By our theorem, the 7 eigenvector of a general $M \in \ker A_f$ are the linear forms in the decomposition of f (up to scalars).

Computing eigenvectors of tensors

In the HPR example, had

$$A_f : S^2V^* \otimes V \xrightarrow{\begin{pmatrix} -C_{f_x} & -C_{f_y} & 0 \\ C_{f_w} & 0 & -C_{f_y} \\ 0 & C_{f_w} & C_{f_x} \end{pmatrix}} V^* \otimes S^2V,$$

with A_f, an 18×18 matrix composed of 6×6 blocks. An element of the kernel can be blocked as (h_1, h_2, h_3), where h_i are quadrics in S^2V^* by viewing $S^2V^* \otimes V$ as $\left(S^2V^* \otimes \langle x \rangle\right) \bigoplus \left(S^2V^* \otimes \langle y \rangle\right) \bigoplus \left(S^2V^* \otimes \langle z \rangle\right)$.

The 2-minors of $\begin{pmatrix} h_1 & h_2 & h_3 \\ x & y & z \end{pmatrix}$ define the locus of eigenvectors.

In the general case the construction is similar: concatenate the (blocked) elements of the kernel with a Koszul matrix and compute the zero set of the minors.

Koszul Algorithm examples: Sylvester Pentahedral

Let $V = \mathbb{C}^4$. The general $f \in S^3V$ has rank 5. The most-square catalecticant is 10×4, so not big enough to detect rank 5.

Koszul flattening: $f \in S^3V \subset V \otimes V \otimes V \leftarrow V \otimes \bigwedge^2 V \otimes V^* \otimes V$

$$\begin{array}{rcl} A_f \colon V^* \otimes \bigwedge^2 V^* & \longrightarrow & V^* \otimes V \\ \mathrm{Hom}(\mathbb{C}^4, \bigwedge^2 \mathbb{C}^4) & \longrightarrow & \mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^4), \end{array}$$

$$A_f = k_2 \otimes C_f, \text{ where } k_2 = \begin{pmatrix} -x & -y & 0 & -z & 0 & 0 \\ w & 0 & -y & 0 & -z & 0 \\ 0 & w & x & 0 & 0 & -z \\ 0 & 0 & 0 & w & x & y \end{pmatrix}.$$

General element of $\mathrm{Hom}(\mathbb{C}^4, \bigwedge^2 \mathbb{C}^4)$ has 5 eigenvectors!

The eigenvectors of a general element of the kernel provide the linear forms in the Waring decomposition.

Koszul Flattening Algorithm

Algorithm

Input $f \in S^d V, V = \mathbb{C}^{n+1}$.

1. *Construct* $A_f \colon \mathrm{Hom}(S^m V, V) \longrightarrow \mathrm{Hom}(\bigwedge^{n-1} V, S^{d-m-1} V)$.
2. *Compute* $\ker A_f$. *Note* $Rank(f) \geq \mathrm{rank}(A_f)/n$.
3. *Set* $Z' =$ *common eigenvectors of a basis of* $\ker A_f$.
   a) *if* $\#Z' = \infty$, *fail.*
   b) *else* $Z' = \{[v_1], \ldots, [v_s]\}$.
4. *Solve* $f = \sum_{i=1}^{s} c_i v_i^d$.

Output: unique Waring decomposition of f.

Success of the Koszul Flattening Algorithm

Here are some effective bounds for the success of our algorithm.

Theorem (O.-Ottaviani'13)

Let $n = 2$, $d = 2m + 1$, $f = \sum_{i=1}^{r} v_i^d$, and set $z_i = [v_i]$, $Z = \{z_1, \dots, z_r\}$. The Koszul Flattening algorithm succeeds when

1. $2r \leq m^2 + 3m + 4$,
2. $2r \leq m^2 + 4m + 2$ *(after finitely many tries).*

and if $n \geq 3$, The Koszul Flattening algorithm succeeds when

1. *n-even, $r \leq \binom{n+m}{n}$ (eigenvectors of $\ker A_f = Z' = Z$),*
2. *n-odd, $r \leq \binom{n+m}{n}$ (e.-vects of $\ker A_f \cap$ e.vects of $(Im(A_f))^{\perp} = Z$),*
3. *$n = 3$, $r \leq \frac{1}{3}(\frac{1}{2}(m+4)(m+3)(m+1) - m^2/2 - m - 8)$*
   ... set $a = 2$ in the algorithm.

General Vector Bundle Method

Consider a line bundle L giving the embedding $X \xrightarrow{|L|} \mathbb{P}\left(H^0(X, L)\right) = \mathbb{P}W$. Let $E \longrightarrow X$ be a vector bundle on X. We get natural maps:

$$\begin{aligned} H^0(E) \otimes H^0(E^* \otimes L) &\longrightarrow H^0(L), \\ H^0(E) \otimes H^0(L)^* &\longrightarrow H^0(E^* \otimes L)^*, \\ H^0(E) &\xrightarrow{A_f} H^0(E^* \otimes L)^*, \end{aligned}$$

where A_f depends linearly on $H^0(L)^*$.

Get the matrix presentation via Koszul matrices when $E = \bigwedge^a Q$, where Q is (at twist of) the quotient bundle on $\mathbb{P}^n$.

Proposition (Landsberg-Ottaviani '12)

Let $f = \sum_{i=1}^r v_i$, and set $z_i = [v_i] \in X \subset \mathbb{P}W$, $Z = \{z_1, \ldots, z_r\}$. Then $H^0(I_Z \otimes E) \subset \ker A_f$, with equality if $H^0(E^* \otimes L) \twoheadrightarrow H^0(E \otimes L_{|Z})$, and $H^0(I_Z \otimes E^* \otimes L) \subset (ImA_f)^\perp$, with equality if $H^0(E) \twoheadrightarrow H^0(E_{|Z})$.

General Vector Bundle Method

Consider a line bundle L giving the embedding $X \xrightarrow{|L|} \mathbb{P}\left(H^0(X, L)\right) = \mathbb{P}W$. Let $E \longrightarrow X$ be a vector bundle on X.

Theorem (O.-Ottaviani'13)

Let $f = \sum_{i=1}^{r} v_i^d$, *and set* $z_i = [v_i] \in X \subset \mathbb{P}W$, $Z = \{z_1, \ldots, z_r\}$. *Assume* $\operatorname{rank}(A_f) = k \cdot Rank(E)$ *and*

$$H^0(I_Z \otimes E) \otimes H^0(I_Z \otimes E^* \otimes L) \longrightarrow H^0(I_Z^2 \otimes L)$$

is surjective.
If X *is not weakly* k*-defective, then the common base locus of* $\ker(A_f)$ *and* $Im(A_f)^{\perp}$ *is given by* Z *(so one can reconstruct* Z *from* f*).*

We use this general result to prove the specific results for each of our algorithms.