Decomposing Tensors into Frames

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Applications of Secant Varieties & Tensors (Join SIAM/(ag)^2)

- Classical Algebraic Geometry: When can a given projective variety $X \subset \mathbb{P}^n$ be isomorphically projected into $\mathbb{P}^{n-1}$?

  Determined by the dimension of the secant variety $\sigma_2(X)$.

- Algebraic Complexity Theory: Bound the border rank of algorithms via equations of secant varieties. Berkeley-Simons program Fall'14

- Algebraic Statistics and Phylogenetics:
  Given contingency tables for DNA of several species, determine the correct statistical model for their evolution.

  Find invariants (equations) of mixture models (secant varieties).

  For star trees / bifurcating trees this is the salmon conjecture.

- Signal Processing: Blind identification of under-determined mixtures, analogous to CDMA technology for cell phones.

  A given signal is the sum of many signals, one for each user.

  Decompose the signal uniquely to recover each user's signal.

- Computer Vision, Neuroscience, Quantum Information Theory, Chemistry...

Symmetric tensor decomposition, a CDMA-like scheme

Many signals (vectors or linear forms):

$$\begin{aligned} \ell_1 &= \ell_{1,1}x_1 + \ell_{1,2}x_2 + \cdots + \ell_{1,n}x_n \\ \ell_2 &= \ell_{2,1}x_1 + \ell_{2,2}x_2 + \cdots + \ell_{2,n}x_n \\ &\vdots \\ \ell_r &= \ell_{r,1}x_1 + \ell_{r,2}x_2 + \cdots + \ell_{r,n}x_n \end{aligned}$$

$\{x_1, \ldots, x_n\}$ — basis of $\mathbb{C}^n$

$\ell_{i,j}$ — scalars

There's no way to recover ℓ_i from the sum $\sum_{i=1}^r \ell_i$.
Instead try to recover ℓ_i from the power-sum $\sum_{i=1}^r \ell_i^d$.

Polynomial:

$$p = \sum_{i=1}^{r} \ell_i^d = \sum_{|I|=n} a_I \binom{n}{I} \cdot x_1^{i_1} \cdot x_2^{i_2} \cdots x_n^{i_n}$$

Symmetric Tensor:

$$(a_I)_I$$

Tensor decomposition:

Recover r and $\ell_{i,j}$ from $(a_I)_I$.

A special Waring decomposition

Consider the following polynomial (symmetric 3×3×3×3-tensor):

$$\begin{aligned} p \;=\; & 59(x_1^4 + x_2^4 + x_3^4) - 16(x_1^3x_2 + x_1x_2^3 + x_1^3x_3 + x_2^3x_3 + x_1x_3^3 + x_2x_3^3) \\ & + 66(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 96(x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2). \end{aligned} \quad (1)$$

A sum of powers representation of p is

$$\tfrac{1}{12}(-5x_1+x_2+x_3)^4+\tfrac{1}{12}(x_1-5x_2+x_3)^4+\tfrac{1}{12}(x_1+x_2-5x_3)^4+\tfrac{1}{12}(3x_1+3x_2+3x_3)^4. \quad (2)$$

The linear forms, appropriately scaled, form a **f**inite **u**nit **n**orm **t**ight **f**rame:

$$V = \frac{1}{3\sqrt{3}}\begin{pmatrix} -5 & 1 & 1 & 3 \\ 1 & -5 & 1 & 3 \\ 1 & 1 & -5 & 3 \end{pmatrix}, \text{ with } VV^T = \frac{4}{3}I_3 \text{ and } ||\mathbf{v_i}|| = 1 \;\forall\, i \quad (3)$$

The title refers to the task of finding the output (2) from the input (1).

This particular decomposition can be found easily using Sylvester's classical *Catalecticant Algorithm*, as explained in [Oeding-Ottaviani '11].
In general, this will be more difficult to do.

Some Frames

\begin{frame}

$$\sqrt{\frac{2}{3}} \cdot \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

\end{frame}

Frames: See [Casazza, et. al], [Cahil-Mixon-Strawn], etc.

A frame is a collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ that span a Hilbert space ($\mathbb{R}^n$ or $\mathbb{C}^n$).

Set $V = \begin{pmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \\ | & | & & | \end{pmatrix}$. We call V a finite unit norm tight frame if

$$V \cdot V^T = \frac{r}{n} \cdot \mathrm{Id}_n \qquad \text{and} \qquad \sum_{j=1}^{n} v_{ij}^2 = 1 \quad \text{for } i = 1, 2, \ldots, r. \tag{4}$$

This is an inhomogeneous system of $n^2 + r$ quadratic equations in $r \cdot n$ unknowns.

The funtf variety, $\mathcal{F}_{r,n}$, is the subvariety of $\mathbb{C}^{r \times n}$ (an affine space) defined by (4).

The frame is called *tight* since for all $\mathbf{x} \in \mathbb{H}$: $\frac{r}{n}||\mathbf{x}||^2 \leq \sum_{i=1}^{r} |\langle \mathbf{x}, \mathbf{v}_i \rangle|^2 \leq \frac{r}{n}||\mathbf{x}||^2$.

The *projective funtf variety* $\mathcal{G}_{r,n}$ is the image of $\mathcal{F}_{r,n}$ in $(\mathbb{P}^{n-1})^r$.

Fundamental Algebraic Geometry of the funtf variety

As you would for any algebraic variety you meet, you should ask the funtf variety:

- Where do you live?
- What is your dimension?
- What is your degree?
- What are your intrinsic defining equations?
- Do you have any friends?
- How are you parametrized?

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As you would for any algebraic variety you meet, you should ask the funtf variety:

- Where do you live? $\mathcal{F}_{r,n} \subset \mathbb{C}^{r \times n}$
- What is your dimension?

Theorem (Dykema-Strawn)

$$\dim(\mathcal{F}_{r,n}) \;=\; (n-1)\cdot(r - \frac{n}{2} - 1) \qquad \textit{provided } \; r > n \geq 2.$$

- What is your degree?
- What are your intrinsic defining equations? $VV^T = \frac{r}{n} \cdot I, \quad ||\mathbf{v}_i|| = 1 \;\; \forall i.$
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Theorem (Cahil-Mixon-Strawn)

$\mathcal{F}_{r,n}$ is irreducible when $r \geq n + 2 > 4$.

- How are you parametrized?

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$\mathcal{F}_{r,n}$ is irreducible when $r \geq n + 2 > 4$.

- How are you parametrized? Great Question!

Numerical Methods can help

r	n	$\dim \mathcal{F}_{r,n}$	$\deg \mathcal{F}_{r,n}$	# components & degrees
3	2	1	$8 \cdot 2$	8 components, each degree 2
4	2	2	$12 \cdot 4$	12 components, each degree 4
5	2	3	112	irreducible
6	2	4	240	irreducible
7	2	5	496	irreducible
4	3	3	$16 \cdot 8$	16 components, each degree 8
5	3	5	1024	irreducible
6	3	7	2048	irreducible
7	3	9	4096	irreducible
5	4	6	$32 \cdot 40$	32 components, each degree 40
6	4	9	20800	irreducible
7	4	12	65536	irreducible

Degree computations performed using `Bertini`.

Frame-Decomposable Tensors

If $T = (t_{i_1 i_2 \cdots i_d})$ is a symmetric tensor in $\mathrm{Sym}_d(\mathbb{C}^n)$ then such a decomposition takes the form

$$T = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i^{\otimes d}. \tag{5}$$

Here $\lambda_i \in \mathbb{C}$ and $\mathbf{v}_i = (v_{i1}, v_{i2}, \ldots, v_{in}) \in \mathbb{C}^n$ for $i = 1, 2, \ldots, r$. The smallest r for which a representation (5) exists is the (Waring) *rank* of T.

A frame decomposition is an expression $T = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i^{\otimes d}$, where $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ form a frame.

The Zariski closure of the set of all tensors T admitting a frame decomposition is an algebraic variety we denote $\mathcal{T}_{r,n,d}$.

When $r = n$ $\mathcal{T}_{r,n,d}$ is the familiar *odeco* variety.
In a similar spirit, we call $\mathcal{T}_{r,n,d}$ the *fradeco* variety.

Fundamental Algebraic Geometry of the fradeco variety

Dear fradeco variety:

- What is your dimension?

Proposition (O-.Robeva–Sturmfels)

For all $r > n$ and $d > 1$, the dimension of $\mathcal{T}_{r,n,d} \subset \mathrm{Sym}_d\mathbb{C}^n$ is bounded above by

$$\min\left\{(n-1)(r-n) + \frac{(n-1)(n-2)}{2} + r - 1\,,\ \binom{n+d-1}{d} - 1\right\}. \quad (6)$$

Notice that $\mathcal{T}_{r,n,d}$ is the closed image of a rational map:

$$\mathcal{F}_{r,n} \times \mathbb{P}^{r-1} \longrightarrow \mathcal{T}_{r,n,d}.$$

The dimension of the image of this map is bounded above by the dimension of the domain.

Conjecture (O-.Robeva–Sturmfels)

The dimension of the variety $\mathcal{T}_{r,n,d}$ is equal to (6) for all $r > n$ and $d > 1$.

Geometric interplay between fradeco and secant varieties

$\sigma_r \nu_d \mathbb{P}^{n-1} :=$ r-th secant variety of the d-th Veronese embedding of $\mathbb{P}^{n-1}$.
lives in $\mathbb{P}(\mathrm{Sym}_d(\mathbb{C}^n))$ and comprises rank r symmetric tensors.
The same ambient space contains the fradeco variety $\mathcal{T}_{r,n,d}$ and all its secant varieties $\sigma_s \mathcal{T}_{r,n,d}$.

Theorem (O-.Robeva–Sturmfels)

For any $r > n \geq d \geq 2$, we have

$$\sigma_{r-n} \nu_d \mathbb{P}^{n-1} \subset \mathcal{T}_{r,n,d} \subset \sigma_r \nu_d \mathbb{P}^{n-1}, \tag{7}$$

and hence $\mathcal{T}_{r-n,n,d} \subset \mathcal{T}_{r,n,d}$ whenever $r \geq 2n$. Also, if $r = r_1 r_2$ with $r_1 \geq 2$ and $r_2 \geq n$, then

$$\sigma_{r_1} \mathcal{T}_{r_2,n,d} \subseteq \mathcal{T}_{r,n,d}. \tag{8}$$

Numerical Answers

Theorem (O-. Robeva–Sturmfels)

The following table gives the degree and some defining polynomials of the fradeco variety $\mathcal{T}_{r,n,d}$ *in all cases when* $n \geq 3$ *and* $1 \leq \dim(\mathcal{T}_{r,n,d}) \cdot \mathrm{codim}(\mathcal{T}_{r,n,d}) \leq 100$*:*

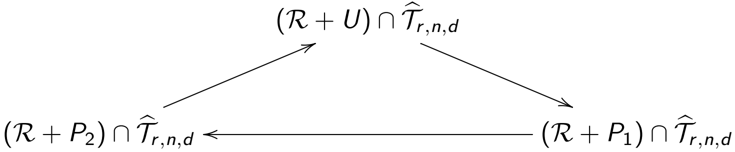
variety	*dim*	*codim*	*degree*	*known equations*
$\mathcal{T}_{4,3,3}$	6	3	17	3 *cubics*, 6 *quartics*
$\mathcal{T}_{4,3,4}$	6	8	74	6 *quadrics*, 37 *cubics*
$\mathcal{T}_{4,3,5}$	6	14	191	27 *quadrics*, 104 *cubics*
$\mathcal{T}_{5,3,4}$	9	5	210	1 *cubic*, 6 *quartics*
$\mathcal{T}_{5,3,5}$	9	11	1479	20 *cubics*, 213 *quartics*
$\mathcal{T}_{6,3,4}$	12	2	99	*none in degree* ≤ 5
$\mathcal{T}_{6,3,5}$	12	8	4269	*one quartic*
$\mathcal{T}_{7,3,5}$	15	5	≥ 38541	*none in degree* ≤ 4
$\mathcal{T}_{8,3,5}$	18	2	690	*none in degree* ≤ 5
$\mathcal{T}_{10,3,6}$	24	3	≥ 16252	*none in degree* ≤ 7
$\mathcal{T}_{5,4,3}$	10	9	830	*none in degree* ≤ 4
$\mathcal{T}_{6,4,3}$	14	5	1860	*none in degree* ≤ 3
$\mathcal{T}_{7,4,3}$	18	1	194	*one in degree* 194

Monodromy for degree calculations (using `Bertini`)

The problem:

`Compute the degree of the image of the map` $\mathcal{F}_{r,n} \times \mathbb{C}^r \longrightarrow \mathrm{Sym}_d\mathbb{C}^n$

- Select random $V \in \mathcal{F}_{r,n}$ and $\lambda \in \mathbb{R}^r$, compute the fradeco tensor $\Sigma_d(V, \lambda)$.
- Fix a random $\mathcal{R} \cong \mathbb{C}^c \subset \mathrm{Sym}_d\mathbb{C}^n$, and point U in the affine space $\mathcal{R} + U$.
- The affine cone $\widehat{\mathcal{T}}_{r,n,d}$ and the affine space $\mathcal{R} + U$ intersect in $\deg(\widehat{\mathcal{T}}_{r,n,d})$ many points in $\mathrm{Sym}_d\mathbb{C}^n$.
- One these points is the known tensor $\Sigma_d(V, \lambda)$.
- Goal: discover all the other intersection points by a `Parameter Homotopy` over the base space $(\mathrm{Sym}_d\mathbb{C}^n)/R$.
- We fix two further random points P_1 and P_2 in $\mathrm{Sym}_d\mathbb{C}^n$.
- The data we fixed now define a (triangular) monodromy loop



- We use `Bertini` to perform each linear parameter homotopy.
- Iterate the process, until we don't find any new points after 20 iterations.

First equations for fradeco varieties: binary forms

Theorem (O-.Robeva–Sturmfels)

Fix $r \in \{3, 4, \ldots, 9\}$. There exists a matrix $\mathcal{M}_r$ with the following properties:

(a) *It has $r-1$ rows and $d-r+1$ columns, entries linear in $t_0, t_1, \ldots, t_d$.*

(b) *The columns involve r of the unknowns t_i and are identical up to index shifts.*

(c) *The maximal minors of $\mathcal{M}_r$ form a Gröbner basis for the prime ideal of $\mathcal{T}_{r,2,d}$.*

These matrices can be chosen as follows:

$$\mathcal{M}_3 = \begin{pmatrix} t_0-3t_2 & t_1-3t_3 & t_2-3t_4 & t_3-3t_5 & \cdots & t_{d-3}-3t_{d-1} \\ 3t_1-t_3 & 3t_2-t_4 & 3t_3-t_5 & 3t_4-t_6 & \cdots & 3t_{d-2}-t_d \end{pmatrix}$$

$$\mathcal{M}_4 = \begin{pmatrix} t_0+t_4 & t_1+t_5 & t_2+t_6 & t_3+t_7 & \cdots & t_{d-4}+t_d \\ t_1-t_3 & t_2-t_4 & t_3-t_5 & t_4-t_6 & \cdots & t_{d-3}+t_{d-1} \\ t_2 & t_3 & t_4 & t_5 & \cdots & t_{d-2} \end{pmatrix}$$

$$\mathcal{M}_5 = \begin{pmatrix} t_0+5t_2 & t_1+5t_3 & t_2+5t_4 & t_3+5t_5 & \cdots & t_{d-5}+5t_{d-3} \\ t_1-3t_3 & t_2-3t_4 & t_3-3t_5 & t_4-3t_6 & \cdots & t_{d-4}-3t_{d-2} \\ 3t_2-t_4 & 3t_3-t_5 & 3t_4-t_6 & 3t_5-t_7 & \cdots & 3t_{d-3}-t_{d-1} \\ 5t_3+t_5 & 5t_4+t_6 & 5t_5+t_7 & 5t_6+t_8 & \cdots & 5t_{d-2}+t_d \end{pmatrix}$$

$$\mathcal{M}_6 = \begin{pmatrix} t_0-t_6 & t_1-t_7 & t_2-t_8 & t_3-t_9 & \cdots & t_{d-6}-t_d \\ t_1+t_5 & t_2+t_6 & t_3+t_7 & t_4+t_8 & \cdots & t_{d-5}+t_{d-1} \\ t_2-t_4 & t_3-t_5 & t_4-t_6 & t_5-t_7 & \cdots & t_{d-4}-t_{d-2} \\ t_3 & t_4 & t_5 & t_6 & \cdots & t_{d-3} \\ t_0+3t_4 & t_1+3t_5 & t_2+3t_6 & t_3+3t_7 & \cdots & t_{d-6}+3t_{d-2} \end{pmatrix} \ldots\ldots$$

First equations for fradeco varieties: Ternary forms

Proposition (O-.Robeva–Sturmfels)

The ideal of the fradeco variety $\mathcal{T}_{4,3,3}$ of ternary cubics of fradeco rank 4 is minimally generated by 3 cubics and 6 quartics.

Proof.

1. `maple`: Find the explicit equations vanishing on $\mathcal{T}_{4,3,3}$ of lowest possible degree using linear algebra and exact arithmetic.
2. `Macaulay2`: This ideal is Cohen-Macaulay of codim. 3 and deg. 17.
3. `Bertini`: $\mathcal{T}_{4,3,3}$ has codim. 3 and deg. 17 to conclude.

□

Actually decomposing tensors into frames

Let $r = 5$ and $d = 8$. We illustrate this method for the binary octic $p =$

$$(-237 - 896\alpha)x^8 + 8(65 + 241\alpha)x^7y + 28(-16 - 68\alpha)x^6y^2 + 56(5 + 31\alpha)x^5y^3$$
$$+70(2 - 56\alpha)x^4y^4 + 56(-7 + 193\alpha)x^3y^5 + 28(32 - 716\alpha)x^2y^6$$
$$+8(-115 + 2671\alpha)xy^7 + (435 - 9968\alpha)y^8,$$

where $\alpha = \sqrt{3} - 2$. We find

$$\mathcal{M}_5 = \begin{pmatrix} -13548\alpha+595 & 3636\alpha-150 & -996\alpha+42 & 348\alpha+18 \\ 2092\alpha-94 & -548\alpha+26 & 100\alpha-22 & 148\alpha+50 \\ -2092\alpha+94 & 548\alpha-26 & -100\alpha+22 & -148\alpha-50 \\ 996\alpha-30 & -348\alpha-6 & 396\alpha+90 & -1236\alpha-317 \end{pmatrix}.$$

This matrix has rank 3 and its left kernel is the span of the vector $\mathbf{w} = (0, 1, 1, 0)$.

Actually decomposing tensors into frames

Therefore, $0 = \mathbf{w} M_5$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} v_{12}^5+5v_{11}^5 & v_{22}^5+5v_{21}^5 & v_{32}^5+5v_{31}^5 & v_{42}^5+5v_{41}^5 & v_{52}^5+5v_{51}^5 \\ v_{11}v_{12}^4-3v_{11}^3v_{12}^2 & v_{21}v_{22}^4-3v_{21}^3v_{22}^2 & v_{31}v_{32}^4-3v_{31}^3v_{32}^2 & v_{41}v_{42}^4-3v_{41}^3v_{42}^2 & v_{51}v_{52}^4-3v_{51}^3v_{52}^2 \\ 3v_{11}^2v_{12}^3-v_{11}^4v_{12} & 3v_{21}^2v_{22}^3-v_{21}^4v_{22} & 3v_{31}^2v_{32}^3-v_{31}^4v_{32} & 3v_{41}^2v_{42}^3-v_{41}^4v_{42} & 3v_{51}^2v_{52}^3-v_{51}^4v_{52} \\ 5v_{11}^3v_{12}^2+v_{11}^5 & 5v_{21}^3v_{22}^2+v_{21}^5 & 5v_{31}^3v_{32}^2+v_{31}^5 & 5v_{41}^3v_{42}^2+v_{41}^5 & 5v_{51}^3v_{52}^2+v_{51}^5 \end{pmatrix}.$$

The 5 columns of the desired tight frame $V = (v_{ij})$ are the distinct zeros in $\mathbb{P}^1$ of

$$f(v_{1i}, v_{2i}) \; = \; v_{1i}v_{2i}^4 - 3v_{1i}^3v_{2i}^2 + 3v_{1i}^2v_{2i}^3 - v_{1i}^4v_{2i} \qquad \text{for } i = 1, \ldots, 5.$$

We find

$$V \; = \; \begin{pmatrix} 1 & 0 & 1 & \alpha & 1 \\ 0 & 1 & 1 & 1 & \alpha \end{pmatrix} \; \in \; \mathcal{G}_{5,2}.$$

It remains to solve the linear system of nine equations in $\lambda = (\lambda_1, \ldots, \lambda_5)$ given by

$$p \; = \; \lambda_1 x^8 + \lambda_2 y^8 + \lambda_3 (x+y)^8 + \lambda_4 (\alpha x + y)^8 + \lambda_5 (x + \alpha y)^8.$$

The unique solution is $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = 1$ and $\lambda_4 = 1552 + 896\sqrt{3}$. $\diamondsuit$

"Waring-enhanced" frame decomposition: ternary quartics

$\sum_{i+j+k=4} \frac{24}{i!j!k!} t_{ijk} x^i y^j z^k = $ $467x^4+152x^3y+1448x^3z+660x^2y^2-1488x^2yz+4020x^2z^2+536xy^3-1992xy^2z$ $+2352xyz^2+944xz^3+227y^4-1000y^3z+2148y^2z^2-1960yz^3+1267z^4.$

Ternary quartics of rank ≤ 5 form a hypersurface of degree 6 in $\mathbb{P}^{14}$ defined by the determinant of the 6×6 catalecticant matrix C.

Here the dimension is one less than expected (Alexander-Hirschowitz Thm.).

The given quartic has a rank 5 catalecticant matrix $C =$

$$\begin{bmatrix} t_{400} & t_{310} & t_{301} & t_{220} & t_{211} & t_{202} \\ t_{310} & t_{220} & t_{211} & t_{130} & t_{121} & t_{112} \\ t_{301} & t_{211} & t_{202} & t_{121} & t_{112} & t_{103} \\ t_{220} & t_{130} & t_{121} & t_{040} & t_{031} & t_{022} \\ t_{211} & t_{121} & t_{112} & t_{031} & t_{022} & t_{013} \\ t_{202} & t_{112} & t_{103} & t_{022} & t_{013} & t_{004} \end{bmatrix} = \begin{bmatrix} 467 & 38 & 362 & 110 & -124 & 670 \\ 38 & 110 & -124 & 134 & -166 & 196 \\ 362 & -124 & 670 & -166 & 196 & 236 \\ 110 & 134 & -166 & 227 & -250 & 358 \\ -124 & -166 & 196 & -250 & 358 & -490 \\ 670 & 196 & 236 & 358 & -490 & 1267 \end{bmatrix}.$$

The kernel of C is spanned by the vector corresponding to the quadric

$$q = 14u^2 - uv - 2uw - 4v^2 - 11vw - 10w^2.$$

The points $(u : v : w)$ in $\mathbb{P}^2$ lying on the conic $\{q = 0\}$ represent all linear forms $ux + vy + wz$ that may appear in a rank 5 decomposition of P.

Our task is to find five points on the conic $\{q = 0\}$ that form a frame $V \in \mathcal{G}_{5,3}$.

This translates into solving a rather challenging system of polynomial equations.

One of the solutions to the system of equations arising from the frame on the conic is

$$V \;=\; (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5) \;=\; \begin{pmatrix} -1 & 2 & 2 & 1+2\sqrt{3} & -1+2\sqrt{3} \\ 2 & 2 & -1 & -2+\sqrt{3} & 2+\sqrt{3} \\ 0 & 1 & -2 & 5 & -5 \end{pmatrix}.$$

The given ternary quartic has the frame decomposition

$$\mathbf{v}_1^{\otimes 4} + \mathbf{v}_2^{\otimes 4} + \mathbf{v}_3^{\otimes 4} + \mathbf{v}_4^{\otimes 4} + \mathbf{v}_5^{\otimes 4}.$$