Defining Equations of Secant Varieties to Segre-Veronese Varieties

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Partially symmetric tensors (in coordinates)

Choose bases $u_1, \ldots, u_m$ of $U^* \cong \mathbb{C}^m$ and $v_1, \ldots, v_n$ of $V^* \cong \mathbb{C}^n$.
In coordinates $T \in U^* \otimes S^2 V^*$ is

$$T = \sum_{i,j,k} T_{ijk} u_i \otimes v_j \otimes v_k$$

with symmetry: $T_{ijk} = T_{ikj}$.
Collect terms:

$$T = \sum_i u_i \otimes \sum_{j,k} T_{ijk} v_j \otimes v_k = \sum_i u_i \otimes A_i$$

with $A_i \in S^2 V^*$, T is a collection of m symmetric $n \times n$ matrices.

Secant varieties and partially symmetric tensors

- Partially symmetric tensors: $T \in U^* \otimes S^2V^*$.
- Segre-Veronese variety: $\operatorname{Seg}(\mathbb{P}U^* \times v_2(\mathbb{P}V^*))$ = projective variety of rank-1 partially symmetric tensors, i.e. tensors of the form $[T] = [u \otimes v \otimes v]$.
- Rank: min r, s.t. $T = \sum_{i=1}^{r} u_i \otimes v_i \otimes v_i$, with $u_i \in U^*$, $v_i \in V^*$.
- Secant variety: $\sigma_r(\operatorname{Seg}(\mathbb{P}U^* \times v_2(\mathbb{P}V^*)))$ = Zariski closure of rank-r partially symmetric tensors.
- T has border rank-r if $[T] \in \sigma_r$ but $[T] \notin \sigma_{r-1}$.
- How do we determine the border rank of a given T?

An appearance in Signal Processing (cf. A. Slapak and A. Yeredor 2010)

- Let $\varphi(\vec{x})$ be a cumulant generating function.
- Construct the (symmetric) Hessian matrix $A_s = \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)(\vec{y}_s)$ for each point $\vec{y}_s$. Sampling φ at m different points, one constructs a partially symmetric tensor $u_1 \otimes A_1 + \cdots + u_m \otimes A_m \in U^* \otimes S^2 V^*$.
- The matrix equations I will describe can be used to study small border ranks of such tensors in the case $m = 3$ and $r \leq 5$.
- When $r \leq 3$ we can also let m be arbitrary.
- A tensor decomposition may be desired. The border rank of this tensor is also useful information.

Equations from flattenings

Realize $T = \sum_i u_i \otimes A_i$ as a linear map (a matrix) via inclusion: $U^* \otimes S^2V^* \subset (U^* \otimes V^*) \otimes V^*$.

$$\psi_{0,T} \colon V \xrightarrow{\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}} U^* \otimes V^*$$

Construction is linear in T:

$$\psi_{0,T} + \psi_{0,T'} = \psi_{0,T+T'}$$

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} + \begin{pmatrix} A_1' \\ A_2' \\ \vdots \\ A_m' \end{pmatrix} = \begin{pmatrix} A_1 + A_1' \\ A_2 + A_2' \\ \vdots \\ A_m + A_m' \end{pmatrix}$$

Equations from flattenings

Realize $T = \sum_i u_i \otimes A_i$ as a linear map (a matrix) via inclusion: $U^* \otimes S^2V^* \subset (U^* \otimes V^*) \otimes V^*$.

$$\psi_{0,T}\colon V \xrightarrow{(\,A_1\ A_2\ \dots\ A_m\,)^t} U^* \otimes V^*$$

Upper bound on rank: Let $A_1 = \begin{pmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \end{pmatrix}$, $A_i = \begin{pmatrix} 0 & \dots & 0 \\ 0 & \ddots & 0 \end{pmatrix}$, $i = 2..m$

$$\psi_{0,T} = \begin{pmatrix} 1 & \dots & 0 & 0 & \cdots & 0 & \dots & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & \dots & 0 & \ddots & 0 \end{pmatrix}^t$$

Let $\kappa_0 = Rank(\psi_{0,T})$. Note: $Rank(T) = 1 \Rightarrow \kappa_0 = 1$:
$\therefore$ subadditivity of rank: $Rank(T) = r \Rightarrow \kappa_0 \leq r$
Let $I_{\kappa_0 \leq r}$ = ideal generated by $(r+1) \times (r+1)$ minors of $\psi_{0,T}$:
$I_{\kappa_0 \leq r}$ are necessary conditions for $brank(T) \leq r$.

Partially symmetric subspace varieties

Definition

*The **subspace variety** $\mathrm{Sub}_{m',n'}$ is the variety of tensors $x \in (U^* \otimes S^2V^*)$ such that there exist vector spaces $\widetilde{U}^* \subset U^*$ and $\widetilde{V}^* \subset V^*$ of dimensions m' and n' respectively with $x \in (\widetilde{U}^* \otimes S^2\widetilde{V}^*) \subset (U^* \otimes S^2V^*)$.*

$T \in \mathrm{Sub}_{m',n'}$ implies that after changing coordinates,

$$\psi_{0,T} = \left(A'_1 \; A'_2 \; \cdots \; A'_{m'} \; 0 \; \cdots \; 0 \right)^t,$$

with $A'_i = \begin{pmatrix} B_i & 0 \\ 0 & 0 \end{pmatrix}$ and B_i a symmetric $n' \times n'$ matrix.
Equations of flattenings define this variety (see next slide).

Partially symmetric subspace varieties

A straightforward adaptation of an argument in [Landsberg-Weyman] using Weyman's geometric technique implies that $\mathrm{Sub}_{m',n'}$ is normal with rational singularities. Moreover

Proposition (Cartwright-Erman-O.)

The defining ideal of $\mathrm{Sub}_{m,n'}$ *equals* $I_{\kappa 0 \leq n'}$ *i.e. the* $(n'+1) \times (n'+1)$ *minors of the flattening* $V \to U^* \otimes V^*$*.*
Moreover, the defining ideal of $\mathrm{Sub}_{m',n'}$ *is the ideal generated by* $I_{\kappa 0 \leq n'}$ *and the* $(m'+1) \times (m'+1)$ *minors of the flattening* $U \to S^2V^*$*.*

Proof involved representation theory, commutative algebra and algebraic geometry.
C. Raicu has recently proved that for any Segre-Veronese variety X, and $r \leq 2$, the ideal $I(X)$ is generated by 3×3 minors of flattenings.

Equations from flattenings

- Have two flattenings $V \to U^* \otimes V^*$ and $U \to S^2V^*$.
- Fact: for $r = 1, 2$, the $(r+1) \times (r+1)$ minors of flattenings are necessary and sufficient conditions for $brank(T) \leq r$.
- Ex.: $T \in \mathbb{C}^3 \otimes S^2\mathbb{C}^4$. Generic border rank is 6.
  - Conditions for $brank(T) \leq 3$?
  $U \to S^2V^*$ is a 3×10 matrix, useless - no 4×4 minors.
  $V \to U^* \otimes V^*$ is a 4×12 matrix, 4×4 minors necessary, but not sufficient.
  - Conditions for $brank(T) \leq 4$?
  - Both flattenings are useless - no 5×5 minors!
- Geometric statement: $\sigma_4\left(\mathbb{P}^2 \times v_2(\mathbb{P}^3)\right) \subsetneq \mathbb{P}^{29}$, but $I_{\kappa_0 \leq 4}$ is trivial.
  - $I_{\kappa_0 \leq r}$ is only non-trivial when $r \leq n = \dim(V)$.
- Need more equations to determine border ranks!

New equations: History

- Symmetric: Aronhold's invariant (1849) is the equation for $\sigma_3(v_3(\mathbb{P}^2)) \subsetneq \mathbb{P}^9$.
- Partially symmetric: E. Toeplitz (1877) essentially gave the equation for $\sigma_5\left(\mathbb{P}^2 \times v_2(\mathbb{P}^3)\right) \subsetneq \mathbb{P}^{29}$.
- Unrestricted: V. Strassen (1983) gave the equation for $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)\right) \subsetneq \mathbb{P}^{26}$
- Ubiquitous: G. Ottaviani (2007) united and generalized all three of these equations with a uniform construction we call exterior flattenings.

Exterior flattenings

Have a natural inclusion $U^* \subset \bigwedge^2 U^* \otimes \bigwedge^1 U$. Construct a new linear map via the inclusion $U^* \otimes S^2V^* \subset \left(V^* \otimes \bigwedge^1 U\right) \otimes \left(V^* \otimes \bigwedge^2 U^*\right)$. Fix $m = 3$. With $T = \sum_i u_i \otimes A_i$, we choose a good basis and write

$$\psi_{1,T} \colon V \otimes \bigwedge^1 U^* \xrightarrow{\begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix}} V^* \otimes \bigwedge^2 U^*.$$

Note $A_i \in S^2V^* \Rightarrow \psi_{1,T}$ is skew-symmetric $\Rightarrow$ rank is even.
Construction is linear in T:

$$\psi_{1,T} + \psi_{1,T'} = \psi_{1,T+T'}$$

$$\begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_3' & -A_2' \\ -A_3' & 0 & A_1' \\ A_2' & -A_1' & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_3+A_3' & -A_2-A_2' \\ -A_3-A_3' & 0 & A_1+A_1' \\ A_2+A_2' & -A_1-A_1' & 0 \end{pmatrix}$$

Exterior Flattenings - new equations for border rank:

Let $\kappa_1 = \kappa_1(T) = Rank(\psi_{1,T})$.

Let $A_1 = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$, $A_i = \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$, $i = 2..m$.

Note: $Rank(T) = 1 \Rightarrow \kappa_1 = 2$:

$$\psi_{1,T} = \begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} & 0 & \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} & \begin{pmatrix} -1 & \cdots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}, & 0 \end{pmatrix}$$

$\therefore$ subadditivity of rank: $Rank(T) = r \Rightarrow \kappa_1 \leq 2r$

Let $I_{\kappa_1 \leq 2r}$ = ideal generated by $2(r+1) \times 2(r+1)$ Pfaffians of $\psi_{1,T}$.

$I_{\kappa_1 \leq 2r}$ are necessary conditions for $brank(T) \leq r$.

Exterior flattenings more generally

Include $U^* \otimes S^2V^* \subset \left(V^* \otimes \bigwedge^j U\right) \otimes \left(V^* \otimes \bigwedge^{j+1} U^*\right)$.

Example $m = 4$: Let $T = \sum_{i=1}^4 u_i \otimes A_i$. In suitable coordinates:

$$\psi_{0,T} \colon V \otimes \textstyle\bigwedge^0 U^* \xrightarrow{\begin{pmatrix} A_1 & A_2 & A_3 & A_4 \end{pmatrix}^t} V^* \otimes \textstyle\bigwedge^1 U^*,$$

$$\psi_{1,T} \colon V \otimes \textstyle\bigwedge^1 U^* \xrightarrow{\begin{pmatrix} 0 & A_3 & -A_2 & 0 \\ -A_3 & 0 & A_1 & 0 \\ A_2 & -A_1 & 0 & 0 \\ A_4 & 0 & 0 & -A_1 \\ 0 & A_4 & 0 & -A_2 \\ 0 & 0 & A_4 & -A_3 \end{pmatrix}} V^* \otimes \textstyle\bigwedge^2 U^*,$$

$$\psi_{2,T} \colon V \otimes \textstyle\bigwedge^2 U^* \xrightarrow{\begin{pmatrix} -A_4 & 0 & 0 & 0 & A_3 & -A_2 \\ 0 & -A_4 & 0 & -A_3 & 0 & A_1 \\ 0 & 0 & -A_4 & A_2 & -A_1 & 0 \\ A_1 & A_2 & A_3 & 0 & 0 & 0 \end{pmatrix}} V^* \otimes \textstyle\bigwedge^3 U^*$$

$$\psi_{3,T} \colon V \otimes \textstyle\bigwedge^3 U^* \xrightarrow{\begin{pmatrix} A_1 & A_2 & A_3 & A_4 \end{pmatrix}} V^* \otimes \textstyle\bigwedge^4 U^*.$$

In general one finds $Rank(T) \leq r \Rightarrow \kappa_j(T) \leq r\binom{m-1}{j}$.

Equations for border rank, $\dim(U) = 3$

Definition

Let $c = (c_0, c_1, c_2)$*.* $I_{\kappa \leq c}$ *= ideal generated by* $I_{\kappa_j \leq c_j}$ *for* $j = 1, 2, 3$*. Define* $\Sigma_{\kappa_i \leq c_i}$ *and* $\Sigma_{\kappa \leq c}$ *to be the subschemes (subvarieties) of* $\mathbb{P}(U^* \otimes S^2 V^*)$ *defined by the ideals* $I_{\kappa_i \leq c_i}$ *and* $I_{\kappa \leq c}$ *respectively.*

Already shown necessary conditions for border rank $\leq r$. Geometric statement:

Proposition

Fix $r \geq 1$*. If* $c = (r, 2r, r)$ *then* $\sigma_r(X) \subseteq \Sigma_{\kappa \leq c}$*.*

We want to know when the necessary conditions are also sufficient.

Main Theorem

Theorem (Cartwright-Erman-O.)

Let $m = \dim(U) = 3$ *and let* $X = \mathrm{Seg}(\mathbb{P}U^* \times v_2(\mathbb{P}V^*))$.
For $r \leq 5$, *the defining ideal of the variety* $\sigma_r(X)$ *is* $I_{\kappa \leq (r,2r,r)}$.

What does it mean?

Restatement for Practical Use

Let $T = u_1 \otimes A_1 + u_2 \otimes A_2 + u_3 \otimes A_3$, A_i *symmetric* $n \times n$ *matrices.*
Suppose $r \leq 5$. *The* $(r+1) \times (r+1)$ *minors of* $\psi_{0,T}$ *and* $2(r+1) \times 2(r+1)$ *Pfaffians of* $\psi_{1,T}$ *are necessary and sufficient conditions to decide the border rank* r *of* T.

Idea of proof:

Theorem (Cartwright-Erman-O.)

For $r \leq 5$, the defining ideal of the variety $\sigma_r(X)$ is $I_{\kappa \leq (r,2r,r)}$.

Proof involves a mixture of commutative algebra, representation theory and algebraic geometry.
We use the result on subspace varieties of $U^* \otimes S^2 V^*$ and prove that the ideal of $\sigma_r(\mathbb{P}^1 \times v_2(\mathbb{P}^{n-1})) = \mathrm{Sub}_{2,r}$ is $I_{\kappa_0 \leq r}$.
Next we handle the case $n = r$ by relating $I_{\kappa_1 \leq 2r}$ to the ideal of the variety of commuting symmetric matrices. This requires the use Erman and Velasco's bound for the dimension of this variety which holds for $r \leq 5$.
Finally, using again a connection to the subspace variety, we reduce the general case to that of the case $n = r$.

Aside: representations for equations

Representation theory allows us to compare equations constructed in different ways via a Schur module description. It also tells us the dimension of the space of equations. Use LiE for experiments.

Proposition (Cartwright-Erman-O.)

As $\mathrm{GL}(U) \times \mathrm{GL}(V)$*-modules we have the following (multiplicity free!):*

$$(I_{\kappa_0 \leq r})_{r+1} = \bigoplus_{|\pi|=r+1} S_\pi U \otimes S_{1^{r+1}+\pi'} V,$$

and when $\dim(U)$ *is* 3*,*

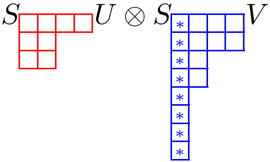
$$(I_{\kappa_1 \leq 2r})_{r+1} = \bigoplus_{|\pi|=r+1} S_\pi U \otimes S_{(3)^{r+1}-\pi'} V,$$

where the partition $\pi = (\pi_1, \pi_2, \pi_3)$ *(at most 3 parts) and* $\pi_3 \geq r+1-n$*.* π' *is the conjugate partition to* π*, and* $S_\pi W$ *=the associated (irreducible)* $\mathrm{GL}(W)$*-module.*

Asside: representations for equations

A representation of polynomials comes from κ_0-conditions if and only if it is of the form:

partition of r+1 $\otimes$ partition transposed with a column adjoined

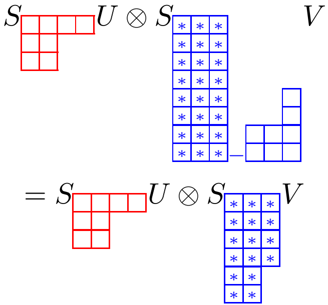


(as long as dimensions allow this)

Asside: representations for equations

In the case $\dim U = 3$, a representation of polynomials comes from κ_1-conditions if and only if it is of the form:

partition of r+1 $\otimes$ 3-wide column with transposed partition excised



(as long as the dimensions allow this)

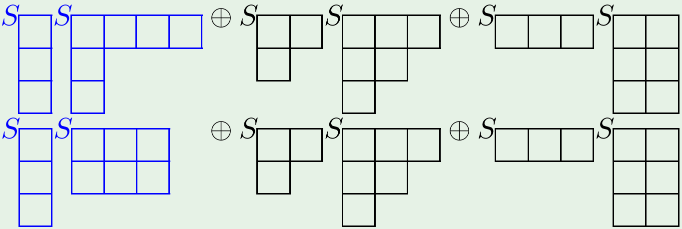
Representation Theory example: $\mathrm{Seg}(\mathbb{P}^2 \times v_2(\mathbb{P}^3))$.

Example (Degree 3)

The 3×3 minors of $U \to S^2V^*$ are included in $(I_{\kappa \leq (2,4,2)})_3$:

$(I_{\kappa_0 \leq 2})_3 \quad =$

$(I_{\kappa_1 \leq 4})_3 \quad =$



The blue modules are the 3×3 minors of $U \to S^2V^*$. Can verify this by computing $\bigwedge^3(U) \otimes \bigwedge(S^2V)$ with Lie:

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> to_part(plethysm([1,1,1],[2,0,0],A3) )
    1X[3,3,0,0] +1X[4,1,1,0]
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Representation Theory example: $\mathrm{Seg}(\mathbb{P}^2 \times v_2(\mathbb{P}^3))$.

Example (Degree 4)

$$(I_{\kappa_0 \leq 3})_4 = S_{(2,2)}S_{(3,3,1,1)} \oplus S_{(2,1,1)}S_{(4,2,1,1)} \oplus S_{(3,1)}S_{(3,2,2,1)} \oplus S_{(4)}S_{(2,2,2,2)}$$

$$(I_{\kappa_1 \leq 6})_4 = S_{(2,2)}S_{(3,3,1,1)} \oplus S_{(2,1,1)}S_{(3,3,2)} \oplus S_{(3,1)}S_{(3,2,2,1)} \oplus S_{(4)}S_{(2,2,2,2)}$$

Note that $\dim((I_{\kappa_0 \leq 3})_4) = \dim((I_{\kappa_1 \leq 6})_4) = 495$.
However $I_{\kappa \leq (3,6,3)} = I_{\kappa_1 \leq 6} + I_{\kappa_0 \leq 3}$ is generated by a 630-dimensional space of quartics. Notice that neither $I_{\kappa_0 \leq 3}$ nor $I_{\kappa_1 \leq 6}$ belongs to the other.

Can also verify these statements using a computer (Macaulay2).

Main Theorem in case $X = \text{Seg}\left(\mathbb{P}^2 \times v_2(\mathbb{P}^3)\right) \subseteq \mathbb{P}^{29}$

- $I(\sigma_2(X))$ is generated by the 3×3 minors of the flattening $\psi_{0,T}$ and by the 3×3 minors of the flattening $U \to S^2V^*$. Now have alternate description; that the 3×3 minors of $\psi_{0,T}$ and the 6×6 principal Pfaffians of $\psi_{1,T}$ also generate the ideal of $\sigma_2(X)$.

- $I(\sigma_3(X))$ is even more interesting (next slide).

- $I(\sigma_4(X)) = I_{\kappa \leq (4,8,4)}$. $I_{\kappa_0 \leq 4}$ and $I_{\kappa_2 \leq 4}$ are trivial.
  $\sigma_4(X)$ is defined by the 10×10 principal Pfaffians of $\psi_{1,T}$.

- $I(\sigma_5(X))$ was previously known: $\sigma_5(X)$ is defective; a hypersurface in $\mathbb{P}^{29}$ defined by the Pfaffian of $\psi_{1,T}$, [Toeplitz] cf. [Ottaviani].

Main Theorem in case $X = \mathrm{Seg}\left(\mathbb{P}^2 \times v_2(\mathbb{P}^3)\right) \subseteq \mathbb{P}^{29}$

- $\sigma_3(X)$ requires minors/Pfaffians from both $\psi_{0,T}$ and $\psi_{1,T}$ (and, unlike the case of $\sigma_2(X)$, the Pfaffians from $\psi_{1,T}$ do not arise from an alternative flattening).
- We saw that $\sigma_3(X)$ is defined by the maximal minors of $\psi_{0,T}$ as well as the 8×8 principal Pfaffians of $\psi_{1,T}$. Neither $I_{\kappa_0 \leq 3}$ nor $I_{\kappa_1 \leq 6}$ is sufficient to generate the ideal of $\sigma_3(X)$.
- In fact, neither $I_{\kappa_0 \leq 3}$ nor $I_{\kappa_1 \leq 6}$ is sufficient to define $\sigma_3(X)$ even set-theoretically. For $I_{\kappa_0 \leq 3}$, this follows from the fact that a generic element $y \in \Sigma_{\kappa_0 \leq 3}$ has $\kappa_1(y) = 8$. For $I_{\kappa_1 \leq 6}$, check that if

$$x := \sum_{i=1}^{3} u_i \otimes (v_1 \otimes v_{i+1} + v_{i+1} \otimes v_1) \in U^* \otimes S^2 V^*,$$

then $\kappa(x) = (4, 6, 4)$, and hence $[x] \in \Sigma_{\kappa_1 \leq 6}$ but $[x] \notin \sigma_3(X)$.

Limits of our equations

- Main theorem says that $I_{(r,2r,r)}$ defines $\sigma_5(\mathbb{P}^2 \times v_2(\mathbb{P}^{n-1}))$ for $r \leq 5$.
- When $r = 6$ we do not know if $I_{(6,12,6)}$ defines $\sigma_6(\mathbb{P}^2 \times v_2(\mathbb{P}^{n-1}))$.
- When $r = 7$, $I_{(7,14,7)}$ does not define $\sigma_7(\mathbb{P}^2 \times v_2(\mathbb{P}^{n-1}))$ even set-theoretically for dimension reasons.
- Note for $r \leq 3$ if we include the equations for $\mathrm{Sub}_{3,n}$ – $(r+1) \times (r+1)$ minors of the flattening $U \to S^2V^*$ – we have defining equations for $\sigma_r(\mathbb{P}^{m-1} \times v_2(\mathbb{P}^{n-1}))$ for all m, n.

How to use these equations in practice

- Given a partially symmetric tensor $T \in U^* \otimes S^2V^*$, to find the border rank r of T with $r \leq 5$ and for $\dim U = 3$ and $\dim V = n$, one needs only check the ranks of the two matrices $\psi_{0,T}$ and $\psi_{1,T}$, a fast computation.
- Recently we have found a tensor decomposition algorithm that uses these equations, (O-,Ottaviani 2011). Ideas are related to Generalized Eigenvectors (Cartwright-Sturmfels 2010) and generalizations of the equations here (Landsberg-Ottaviani 2011).

Thanks!