Secant Varieties and Equations of Abo-Wan Hypersurfaces

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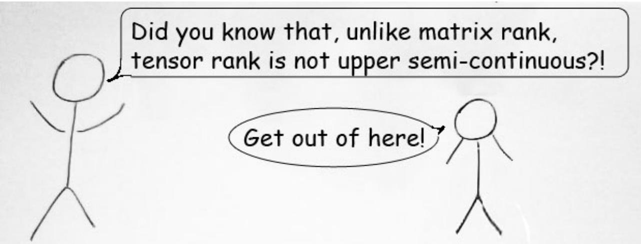
Secant varieties

Suppose X is an algebraic variety in $\mathbb{P}^N$.

The *X-rank* of $[p] \in \mathbb{P}^N$ is the min. r such that $p = \sum_{i=1}^{r} x_i$ with $[x_i] \in X$.

The Zariski closure of the points of X-rank r is the *r-secant variety to X*, denoted $\sigma_r(X)$, and consists of the points in $\mathbb{P}^N$ of *X-border rank r*.

Taking the Zariski closure often causes problems.



If $p = b\otimes a\otimes a + a\otimes b\otimes a + a\otimes a\otimes b$, with a, b independent,
$\mathrm{Rank}(p) = 3$ but $\mathrm{Brank}(p) = 2$!

Secant varieties and tensors

Let $A = \{a_i\}, B = \{b_j\}, C = \{c_k\}$, be $\mathbb{C}$-vector spaces, then the tensor product $A \otimes B \otimes C$ has basis elements of the form $a_i \otimes b_j \otimes c_k$, with coordinates p_{ijk}.

- Segre variety (rank 1 tensors): (Independence model) Defined by

$$\begin{array}{rcl} \mathrm{Seg} : \mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C & \longrightarrow & \mathbb{P}\big(A \otimes B \otimes C\big) \\ ([u], [v], [w]) & \longmapsto & [u \otimes v \otimes w]. \end{array}$$

  In coordinates: $p_{i,j,k} = u_i v_j w_k$.

- The r^{th} secant variety of a variety $X \subset \mathbb{P}^n$: (Mixture model)

$$\sigma_r(X) := \overline{\bigcup_{x_1,\dots,x_r \in X} \mathbb{P}(\mathrm{span}\{x_1, \dots, x_r\})} \subset \mathbb{P}^n.$$

General points of $\sigma_r(\mathrm{Seg}(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C))$ have the form $[\sum_{s=1}^r u^s \otimes v^s \otimes w^s]$, or in coordinates: $p_{i,j,k} = \sum_{s=1}^r u_i^s v_j^s w_k^s$.

(*Might also work over $\mathbb{R}$ or Δ-probability simplex, but not today.)

Some Applications of Secant Varieties

- Classical Algebraic Geometry: When can a given projective variety $X \subset \mathbb{P}^n$ be isomorphically projected into $\mathbb{P}^{n-1}$?

  Determined by the dimension of the secant variety $\sigma_2(X)$.

- Algebraic Complexity Theory: Bound the border rank of algorithms via equations of secant varieties. Berkeley-Simons program Fall'14

- Algebraic Statistics and Phylogenetics:
  Given contingency tables for DNA of several species, determine the correct statistical model for their evolution.

  Find invariants (equations) of mixture models (secant varieties).

  For star trees / bifurcating trees this is the salmon conjecture.

- Signal Processing: Blind identification of under-determined mixtures, analogous to CDMA technology for cell phones.

  A given signal is the sum of many signals, one for each user.

  Decompose the signal uniquely to recover each user's signal.

- Computer Vision, Neuroscience, Quantum Information Theory, Chemistry...

First questions for secant varieties

Given $X \subset \mathbb{P}^N$ we ask:

1. [Dimensions] What is the dimension of $\sigma_r(X)$?
   – When does $\sigma_r(X)$ fill the ambient $\mathbb{P}^N$? (defectivity)
2. [Equations] What are the polynomial defining equations of $\sigma_r(X)$?
3. [Generic Identifiability] For *generic* $x \in \mathbb{P}^N$, does x have a unique expression as a sum of points from X? (ignoring trivialities)
4. [Decomposition] For my favorite $x \in \mathbb{P}^N$, can you find an expression of x as a sum of points from X?

Sometimes $2 \Rightarrow 1 \Rightarrow 3 \Rightarrow 4$

Tensors with different types of symmetry

Suppose A, B, C are vector spaces over $\mathbb{C}$. Hypermatrices, symmetric, partially symmetric, skew-symmetric, and partially skew-symmetric tensors:

Space	Hypermatrix	Symmetry
$A \otimes B \otimes C$	$(T_{i,j,k})$	
$S^d A$	$(T_{i_1,\dots,i_d})$	$T_{i_1,\dots,i_d} = T_{\sigma(i_1),\dots,\sigma(i_d)}$ for all $\sigma \in \mathfrak{S}_d$
$A \otimes S^2 B$	$(T_{i,j,k})$	$T_{i,j,k} = T_{i,k,j}$ for all i, j, k
$\bigwedge^{k+1} A$	$(T_{i_0,\dots,i_k})$	$T_{i_0,\dots,i_k} = sgn(\sigma) T_{\sigma(i_0),\dots,\sigma(i_k)}$ for all $\sigma \in \mathfrak{S}_{k+1}$
$A \otimes \bigwedge^{k+1} B$	$(T_{i,j_0,\dots,j_k})$	$T_{i,j_0,\dots,j_k} = sgn(\sigma) T_{i,\sigma(j_0),\dots,\sigma(j_k)}$ for all $\sigma \in \mathfrak{S}_{k+1}$

Partially skew-symmetric tensors (in coordinates)

Choose bases $u_1, \ldots, u_m$ of $A \cong \mathbb{C}^m$ and $v_1, \ldots, v_n$ of $B \cong \mathbb{C}^n$.
In coordinates $T \in A \otimes \bigwedge^2 B$ is

$$T = \sum_{i,j,k} T_{ijk} u_i \otimes v_j \otimes v_k$$

with symmetry: $T_{ijk} = -T_{ikj}$.
Collect terms:

$$T = \sum_i u_i \otimes \sum_{j,k} T_{ijk} v_j \otimes v_k = \sum_i u_i \otimes T_i$$

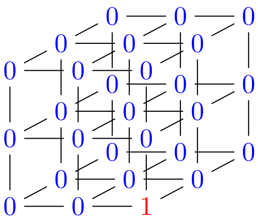
with $T_i \in \bigwedge^2 B$, T is a collection of m skew-symmetric $n \times n$ matrices.

Tensors and examples of classical algebraic varieties

Consider each symmetry type and the (classical) variety of "rank-1" tensors:

Name	Notation	Points	Ambient Space
Segre	$\mathrm{Seg}\,(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$	$[a \otimes b \otimes c]$	$\mathbb{P}(A \otimes B \otimes C)$
Veronese	$\nu_d(\mathbb{P}A)$	$[a \otimes \cdots \otimes a]$	$\mathbb{P}(S^d A)$
Segre-Veronese	$\mathrm{Seg}\,(\mathbb{P}A \times \nu_2(\mathbb{P}V))$	$[a \otimes b \otimes b]$	$\mathbb{P}(A \otimes S^2 B)$
Grassmannian	$\mathbb{G}(k, A)$	$[a_0 \wedge \cdots \wedge a_k]$	$\mathbb{P}\left(\bigwedge^{k+1} A\right)$
Segre-Grassmann	$\mathrm{Seg}(\mathbb{P}A \times \mathbb{G}(k, B))$	$[a \otimes (b_0 \wedge \cdots \wedge b_k)]$	$\mathbb{P}(A \otimes \bigwedge^{k+1} B)$

In each case a rank 1 tensor can be put in the form (in an appropriate basis):



How to email the Segre-Grassmann $\mathrm{Seg}(\mathbb{P}A \times \mathbb{G}(k, B))$

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KK = QQ;
makeRank1 = (m,k,n)->(
     E = random(KK^(m+1),KK^(n+1));
     e = gens minors(k+1,E);
     v = random(KK^(m+1),KK^1);
     matrix {flatten entries(v*e)}
)
makeRank1(2,2,5) -- a point on Seg(P^2 x G(2,5))
sum(5,i->makeRank1(2,2,5)) -- a point on sig_5(Seg(P^2 x G(2,5)))
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(In practice I'm more careful to name the coordinates and avoid collisions.)
Let $v = (v_0, \dots, v_m)$, and let $E = (e_{i,j})$ be a $(k+1) \times (n+1)$ matrix.
Get a $(m+1) \times \binom{n+1}{k+1}$ vector for a point on $\mathrm{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))$ as

$$(v_i \cdot \Delta_I(E))_{i,I} \,,$$

where Δ_I – maximal minor of E with columns $I = (i_1, \dots, i_{k+1})$.
Pseudo-random points on $\sigma_r(\mathrm{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n)))$: let v and E have random entries, and summing r pseudorandom points of $\mathrm{Seg}(\mathbb{P}^m \times \mathbb{G}(k, n))$.

What is a flattening?

Express a tensor $T = \sum_{i,j,k} p_{ijk} a_i \otimes b_j \otimes c_k \in A \otimes B \otimes C$ as a matrix (in 3 ways):

$$\begin{aligned} T &= \sum_i a_i \otimes \left(\sum_{j,k} p_{ijk} b_j \otimes c_k \right) &\in A \otimes (B \otimes C), \\ T &= \sum_j b_j \otimes \left(\sum_{i,k} p_{ijk} a_i \otimes c_k \right) &\in B \otimes (A \otimes C), \\ T &= \sum_k \left(\sum_{i,j} p_{ijk} a_i \otimes b_j \right) \otimes c_k &\in (A \otimes B) \otimes C. \end{aligned}$$

Example: $T = [p_{ijk}] \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ to $\mathbb{C}^3 \otimes (\mathbb{C}^3 \otimes \mathbb{C}^3) \cong \mathbb{C}^3 \otimes \mathbb{C}^9$ flattens to:

$$\psi_{0,T} = \left(\begin{array}{ccc|ccc|ccc} p_{111} & p_{121} & p_{131} & p_{112} & p_{122} & p_{132} & p_{113} & p_{123} & p_{133} \\ p_{211} & p_{221} & p_{231} & p_{212} & p_{222} & p_{232} & p_{213} & p_{223} & p_{233} \\ p_{311} & p_{321} & p_{331} & p_{312} & p_{322} & p_{332} & p_{313} & p_{323} & p_{333} \end{array} \right)$$

When they exist, $(r+1) \times (r+1)$ minors of $\psi_{0,T}$ are (some) equations of $\sigma_r(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$.

Equations from flattenings

Realize $T \in A \otimes B \otimes C = A \otimes (B \otimes C)$ as a linear map (a matrix)

$$\psi_{0,T} \colon A^* \xrightarrow{(\, T_1 \ T_2 \ \dots \ T_l \,)^t} B \otimes C$$

Rank 1 case: Let $T_1 = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$, $T_i = \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$, $i = 2 \dots m$

$$\psi_{0,T} = \begin{pmatrix} 1 & \dots & 0 & 0 & \cdots & 0 & \dots & 0 & \cdots & 0 \\ & \ddots & & & \ddots & & & & \ddots & \\ 0 & & 0 & 0 & & 0 & \dots & 0 & & 0 \end{pmatrix}^t$$

If $\mathrm{Rank}(T) = 1$, then $\mathrm{Rank}(\psi_{0,T}) = 1$. Construction is linear in T:

$$\psi_{0,T} + \psi_{0,T'} = \psi_{0,T+T'}$$

$$\begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_l \end{pmatrix} + \begin{pmatrix} T_1' \\ T_2' \\ \vdots \\ T_l' \end{pmatrix} = \begin{pmatrix} T_1 + T_1' \\ T_2 + T_2' \\ \vdots \\ T_l + T_l' \end{pmatrix}$$

$\mathrm{Rank}(\psi_{0,T}) + \mathrm{Rank}(\psi_{0,T'}) \leq \mathrm{Rank}(\psi_{0,T} + \psi_{0,T'}) = \psi_{0,T+T'}$

By subadditivity of rank, if $\mathrm{Rank}(T) = r$, then $\mathrm{Rank}(\psi_{0,T}) \leq r$.

The $(r+1) \times (r+1)$ minors of $\psi_{0,T}$ are necessary conditions for $\mathrm{Brank}(T) \leq r$.

However, flattenings are trivial when $\mathrm{Rank}(T) > \min\{\dim A, \dim B, \dim C\}$

Symmetric Flattenings

Consider $\phi \in Sym^d V$ as a symmetric multilinear form:
ϕ eats d vectors (symmetrically) and spits out a number.

If we only feed s vectors to ϕ, it still wants to eat $d-s$ more.
So we can construct a linear map

$$\begin{aligned} \phi_{s,d-s} : Sym^s(V^*) &\to Sym^{d-s}V \\ [v_1, \ldots, v_s] &\mapsto \phi(v_1, \ldots, v_s, \_, \ldots, \_) \end{aligned}$$

Macaulay (1916) showed that $\operatorname{Brank} \phi \geq \operatorname{Rank} \phi_{s,d-s}$ for all $1 \leq s \leq d$.

The minors of $\phi_{s,d-s}$ are called minors of *Catalecticants*.
(Also called *symmetric flattenings* or *special Hankel matrices*).

Give some equations for the secant varieties to Veronese varieties.

Flattenings and the Segre variety

Classical: the ideal of any Segre is generated by all 2×2 minors of flattenings.

Theorem (Raicu (2012), (Garcia, Stillman and Sturmfels Conj.))

The prime ideal of $\sigma_2(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n))$ *is generated by the* 3×3 *minors of flattenings.*

Built on work of Landsberg-Manivel, Landsberg-Weyman, Geramita et al., Allman-Rhodes.

Raicu also proved the stronger analogous result for the secant variety of any Segre-Veronese.

[Michalek-O.-Zwiernik (2014)] gave a toric proof of the scheme-theoretic version that works in any characteristic using "secant cumulants."

Flattenings run out quickly: $\sigma_3(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$ has no equations from flattenings since there are no 4×4 minors of 3×9, ([Strassen'83, Y. Qi'14].)

Exterior flattenings

Suppose $T \in A \otimes B \otimes C$. Have a natural inclusion $A \subset \bigwedge^2 A \otimes A^*$.
Construct a new linear map via the inclusion

$$A \otimes B \otimes C \quad \subset \quad \bigwedge^2 A \otimes A^* \otimes B \otimes C \quad = \quad (A^* \otimes B) \otimes \left(\bigwedge^2 A \otimes C \right).$$

Fix $\dim(A) = 3$. With $T = \sum_i u_i \otimes T_i$, we choose a good basis and write

$$\psi_{1,T} \colon A \otimes B^* \xrightarrow{\begin{pmatrix} 0 & T_3 & -T_2 \\ -T_3 & 0 & T_1 \\ T_2 & -T_1 & 0 \end{pmatrix}} \bigwedge^2 A \otimes C.$$

Basic idea:

$\psi_{1,T+T'} = \psi_T + \psi_{T'}$	construction is linear in T
$\operatorname{Rank}(T) = 1 \Rightarrow \operatorname{Rank}(\psi_T) = 2$	base case
$\therefore \operatorname{Rank}(T) = r \Rightarrow \operatorname{Rank}(\psi_T) \leq 2r$	subadditivity of matrix rank

The $(2r+1) \times (2r+1)$ minors of $\psi_{1,T}$ are necessary conditions for $\operatorname{Brank}(T) \leq r$.

Exterior Flattenings: $\text{Rank}(T) = 1 \Rightarrow \text{Rank}\,\psi_{1,T} = 2$

Let $T_1 = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$, $T_i = \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$, $i = 2..m$.

$$\psi_{1,T} = \begin{pmatrix} 0 & T_3 & -T_2 \\ -T_3 & 0 & T_1 \\ T_2 & -T_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} & \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} & 0 & \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} & \begin{pmatrix} -1 & \cdots & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}, & 0 \end{pmatrix}$$

Result is invariant under natural changes of coordinates.

Exterior flattenings (partially symmetric case)

Have natural inclusions $A \subset \bigwedge^2 A \otimes A^*$, $S^2 B \subset B \otimes B$.
Construct a new linear map via the inclusion $A \otimes S^2 B \subset (B \otimes A^*) \otimes \left(B \otimes \bigwedge^2 A\right)$.
Fix $m = 3$. With $T = \sum_i u_i \otimes T_i$, we choose a good basis and write

$$\psi_{1,T}\colon B^* \otimes A \xrightarrow{\begin{pmatrix} 0 & T_3 & -T_2 \\ -T_3 & 0 & T_1 \\ T_2 & -T_1 & 0 \end{pmatrix}} B \otimes \bigwedge^2 A.$$

Note $T_i \in S^2 B \Rightarrow \psi_{1,T}$ is skew-symmetric $\Rightarrow$ rank is even.
Construction is linear in T:

$$\psi_{1,T} + \psi_{1,T'} = \psi_{1,T+T'}$$

$$\begin{pmatrix} 0 & T_3 & -T_2 \\ -T_3 & 0 & T_1 \\ T_2 & -T_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & T_3' & -T_2' \\ -T_3' & 0 & T_1' \\ T_2' & -T_1' & 0 \end{pmatrix} = \begin{pmatrix} 0 & T_3+T_3' & -T_2-T_2' \\ -T_3-T_3' & 0 & T_1+T_1' \\ T_2+T_2' & -T_1-T_1' & 0 \end{pmatrix}$$

$2(r+1) \times 2(r+1)$ Pfaffians of $\psi_{1,T}$ are necessary conditions for $\mathrm{Brank}(T) \leq r$.

Exterior flattenings more generally

Include $A \otimes B \otimes C \subset \left(B \otimes \bigwedge^j A^*\right) \otimes \left(C \otimes \bigwedge^{j+1} A\right)$.

Example $\dim(A) = l = 4$: Let $T = \sum_{i=1}^4 u_i \otimes T_i$. In suitable coordinates:

$$\psi_{0,T}\colon B^* \otimes \bigwedge^0 A^* \xrightarrow{(\,T_1\ T_2\ T_3\ T_4\,)^t} C \otimes \bigwedge^1 A,$$

$$\psi_{1,T}\colon B^* \otimes \bigwedge^1 A^* \xrightarrow{\begin{pmatrix} 0 & T_3 & -T_2 & 0 \\ -T_3 & 0 & T_1 & 0 \\ T_2 & -T_1 & 0 & 0 \\ T_4 & 0 & 0 & -T_1 \\ 0 & T_4 & 0 & -T_2 \\ 0 & 0 & T_4 & -T_3 \end{pmatrix}} C \otimes \bigwedge^2 A,$$

$$\psi_{2,T}\colon B^* \otimes \bigwedge^2 A^* \xrightarrow{\begin{pmatrix} -T_4 & 0 & 0 & 0 & T_3 & -T_2 \\ 0 & -T_4 & 0 & -T_3 & 0 & T_1 \\ 0 & 0 & -T_4 & T_2 & -T_1 & 0 \\ T_1 & T_2 & T_3 & 0 & 0 & 0 \end{pmatrix}} C \otimes \bigwedge^3 A$$

$$\psi_{3,T}\colon B^* \otimes \bigwedge^3 A^* \xrightarrow{(\,T_1\ T_2\ T_3\ T_4\,)} C \otimes \bigwedge^4 A.$$

In general one finds $\mathrm{Rank}(T) \leq r \Rightarrow \mathrm{Rank}(\psi_{j,T}) \leq r\binom{l-1}{j}$.
We want to know when the necessary conditions are also sufficient.

Subspace varieties

Definition

The **subspace variety** $\mathrm{Sub}_{p,q,r}(A\otimes B\otimes C)$ *is the variety of tensors* $[T] \in \mathbb{P}(A\otimes B\otimes C)$ *such that there exist subspaces* $\mathbb{C}^p \subseteq A, \mathbb{C}^q \subseteq B, \mathbb{C}^r \subseteq C$, *and* $[T] \in \mathbb{P}(\mathbb{C}^p\otimes\mathbb{C}^q\otimes\mathbb{C}^r)$.

Theorem (Thm. 3.1, Landsberg–Weyman '07)

$\mathrm{Sub}_{p,q,r}(A\otimes B\otimes C)$ *is normal with rational singularities. Its ideal is generated by the minors of flattenings;*

$$\left(\bigwedge^{p+1} A^*\otimes\bigwedge^{p+1}(B^*\otimes C^*)\right) \oplus \left(\bigwedge^{q+1} B^*\otimes\bigwedge^{q+1}(A^*\otimes C^*)\right) \oplus \left(\bigwedge^{r+1}(A^*\otimes B^*)\otimes\bigwedge^{r+1} C^*\right)$$

Key Point: $\mathrm{Sub}_{r,r,r} \supseteq \sigma_r(\mathbb{P}A \times \mathbb{P}B \times \mathbb{P}C)$ and therefore get some (determinantal) equations of the secant varieties.

Note: C. Raicu has recently proved that for any Segre-Veronese variety X, and $r \leq 2$, the ideal $I(\sigma_2(X))$ is generated by 3×3 minors of flattenings.

Partially symmetric subspace and secant varieties

Definition

The subspace variety $\mathrm{Sub}_{p,q}(A \otimes S^2 B)$ *is the variety of tensors* $[T] \in \mathbb{P}(A \otimes S^2 B)$ *such that there exist subspaces* $\mathbb{C}^p \subseteq A, \mathbb{C}^q \subseteq B$, *and* $[T] \in \mathbb{P}(\mathbb{C}^p \otimes S^2 \mathbb{C}^q)$.

$T \in \mathrm{Sub}_{p,q}(A \otimes S^2 B)$ implies that after changing coordinates,

$$\psi_{0,T} = \left(T'_1 \; T'_2 \; \cdots \; T'_{m'} \; 0 \cdots \; 0 \right)^t,$$

with $T'_i = \left(\begin{smallmatrix} B_i & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and B_i a symmetric $q \times q$ matrix.

Proposition (Cartwright-Erman-O. (2012))

The defining ideal of $\mathrm{Sub}_{p,q}(A \otimes S^2 B)$ *is generated by the* $(p+1) \times (p+1)$ *minors of the flattening* $B^* \to A \otimes B$ *and the* $(q+1) \times (q+1)$ *minors of the flattening* $A^* \to S^2 B$.

Theorem (Cartwright-Erman-O. (2012))

For $r \leq 5$, *the ideal of* $\sigma_r(\mathrm{Seg}(\mathbb{P}^2 \times \nu_2(\mathbb{P}B)))$ *is generated by* $(r+1) \times (r+1)$ *minors of flattenings, and* $(2r+2) \times (2r+2)$ *Pfaffians of exterior flattenings.*

Skew-symmetric subspace varieties

Definition

The **subspace variety** $\mathrm{Sub}_p(\bigwedge^{k+1} A)$ *is the variety of tensors* $[T] \in \mathbb{P}\left(\bigwedge^{k+1} A\right)$ *such that there exist a subspace* $\mathbb{C}^p \subseteq A$*, and* $[T] \in \mathbb{P}\left(\bigwedge^{k+1} \mathbb{C}^p\right)$.

The equations of this variety are mysterious when $k \geq 2$:

Proposition (Boralevi-O.(2012)[1])

The ideal of the subspace variety $\mathrm{Sub}_5(\bigwedge^3 \mathbb{C}^7) \subset \mathbb{P}^{34}$ *is generated in degree 3 by the* GL_7*-modules* $S_{(3,1,1,1,1,1,1)}\mathbb{C}^7$ *(28 cubics) and* $S_{(2,2,2,1,1,1)}\mathbb{C}^7$ *(224 cubics).*

The space of 28 cubics is vector space isomorphic to quadrics on 7 variables. The space of 224 cubics is inherited from a space of 20 cubics on 6 variables, which is vector space isomorphic the span of the 3×3 minors of a 3×6 matrix.

Used Weyman's "Geometric Technique," Representation Theory, Bott's algorithm, and careful combinatorial book-keeping.

[1] Appeared in [J.M. Landsberg, Tensors: Geometry and Applications, (p405-407), AMS GSM, vol. 128, (2012)].

Equations of secant varieties via more general flattenings

- Symmetric: Aronhold's invariant (1849): $\sigma_3(v_3(\mathbb{P}^2)) \subsetneq \mathbb{P}^9$.
- Partially symmetric: Toeplitz (1877): $\sigma_5\left(\mathbb{P}^2 \times \nu_2(\mathbb{P}^3)\right) \subsetneq \mathbb{P}^{29}$.

  Cartwright-Erman-O. (2012): $\sigma_r\left(\mathbb{P}^2 \times \nu_2(\mathbb{P}^n)\right) \subsetneq \mathbb{P}^{3\binom{n+2}{2}-1}$, $r \leq 5$.
- Partially skew-symmetric: Abo-Wan (2013): $\sigma_{3\ell}(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell+2))) \subsetneq \mathbb{P}^{3\binom{4\ell+3}{2}-1}$.
- Unrestricted: Strassen (1983): $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)\right) \subsetneq \mathbb{P}^{26}$
- Ubiquitous: Ottaviani (2007) united and generalized all of these equations with a uniform construction we call exterior flattenings.
- Landsberg-Ottaviani (2012): Many more cases, much more general.

Abo-Wan Hypersurfaces: $\sigma_{3\ell+2}(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1, 4\ell+2)))$

Consider $\sigma_5(\mathbb{P}^2 \times \mathbb{G}(1,6)) \subset \left(\mathbb{P}\mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^7\right) = \mathbb{P}(V \otimes \bigwedge^2 W)$.

View $T \in V \otimes \bigwedge^2 W)$ as an element in $\bigwedge^2 V^* \otimes \bigwedge^2 W$, which induces

$$\varphi_T \colon V \otimes W^* \to V^* \otimes W,$$

Explicitly φ_T is the (21×21) Kronecker product of two matrices:

$$\varphi_T = \begin{pmatrix} 0 & v_1 & -v_2 \\ -v_1 & 0 & v_3 \\ v_2 & -v_3 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} & e_{17} \\ -e_{12} & 0 & e_{23} & e_{24} & e_{25} & e_{26} & e_{27} \\ -e_{13} & -e_{23} & 0 & e_{34} & e_{35} & e_{36} & e_{37} \\ -e_{14} & -e_{24} & -e_{34} & 0 & e_{45} & e_{46} & e_{47} \\ -e_{15} & -e_{25} & -e_{35} & -e_{45} & 0 & e_{56} & e_{57} \\ -e_{16} & -e_{26} & -e_{36} & -e_{46} & -e_{56} & 0 & e_{67} \\ -e_{17} & -e_{27} & -e_{37} & -e_{47} & -e_{57} & -e_{67} & 0 \end{pmatrix}$$

Replace $v_i \otimes e_{jk}$ with x_{ijk} we obtain the matrix $\varphi_T =$ (too big for screen).

Check: $\mathrm{Rank}(T) = 1 \Rightarrow \mathrm{Rank}\, \varphi_T = 4$ so $\mathrm{Rank}(T) = r \Rightarrow \mathrm{Rank}\, \varphi_T \leq 4r$.

In particular if $\mathrm{Rank}\, T = 5$ then $\mathrm{Rank}\, \varphi_T \leq 20$, so $\det \varphi_T$ vanishes.

Check: $\mathrm{Rank}(\varphi_T) = 21$ for random T, so $\det \varphi_T$ is non-trivial.
$\det \varphi_T$ is an equation of $\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1,6)))$. (Verified in Macaulay 2.)

How to email an exterior flattening

```
KK = ZZ/101
T = KK[v1, v2, v3]**KK[w12, w13, w14, w15, w16, w17, w23, w24, w25,
Mv = matrix {{0, v1, -v2}, {-v1, 0, v3}, {v2, -v3,0}}
Mw = matrix {{   0,  w12,  w13,  w14,  w15,  w16,  w17},
             {-w12,    0,  w23,  w24,  w25,  w26,  w27} ,
             {-w13, -w23,    0,  w34,  w35,  w36,  w37},
             {-w14, -w24, -w34,    0,  w45,  w46,  w47},
             {-w15, -w25, -w35, -w45,    0,  w56,  w57},
             {-w16, -w26, -w36, -w46, -w56,    0,  w67},
             {-w17, -w27, -w37, -w47, -w57, -w67, 0   }
K = Mw**Mv
for i from 1 to 6 do print(i, rank diff(K, sum(i, j-> makeRank1())))
R = KK[x112, x212, x312, x113, x213, x313, x114, x214, x314, x115, x
P = T**R
vwTOx = v1*w12*x112 + v1*w13*x113 + v1*w14*x114 + v1*w15*x115 + v1*w
myMat = diff(sub(K,P),vwTox)
mypoly = det(myMat, Strategy => Cofactor);
```

Kronecker product

$$
\left(\begin{array}{ccccccccccccccc}
0 & 0 & 0 & 0 & x_{112} & -x_{212} & 0 & x_{113} & -x_{213} & 0 & x_{114} & -x_{214} & 0 & x_{115} & -x_{215} \\
0 & 0 & 0 & -x_{112} & 0 & x_{312} & -x_{113} & 0 & x_{313} & -x_{114} & 0 & x_{314} & -x_{115} & 0 & x_{315} \\
0 & 0 & 0 & x_{212} & -x_{312} & 0 & x_{213} & -x_{313} & 0 & x_{214} & -x_{314} & 0 & x_{215} & -x_{315} & 0 \\
0 & -x_{112} & x_{212} & 0 & 0 & 0 & 0 & x_{123} & -x_{223} & 0 & x_{124} & -x_{224} & 0 & x_{125} & -x_{225} \\
x_{112} & 0 & -x_{312} & 0 & 0 & 0 & -x_{123} & 0 & x_{323} & -x_{124} & 0 & x_{324} & -x_{125} & 0 & x_{325} \\
-x_{212} & x_{312} & 0 & 0 & 0 & 0 & x_{223} & -x_{323} & 0 & x_{224} & -x_{324} & 0 & x_{225} & -x_{325} & 0 \\
0 & -x_{113} & x_{213} & 0 & -x_{123} & x_{223} & 0 & 0 & 0 & 0 & x_{134} & -x_{234} & 0 & x_{135} & -x_{235} \\
x_{113} & 0 & -x_{313} & x_{123} & 0 & -x_{323} & 0 & 0 & 0 & -x_{134} & 0 & x_{334} & -x_{135} & 0 & x_{335} \\
-x_{213} & x_{313} & 0 & -x_{223} & x_{323} & 0 & 0 & 0 & 0 & x_{234} & -x_{334} & 0 & x_{235} & -x_{335} & 0 \\
0 & -x_{114} & x_{214} & 0 & -x_{124} & x_{224} & 0 & -x_{134} & x_{234} & 0 & 0 & 0 & 0 & x_{145} & -x_{245} \\
x_{114} & 0 & -x_{314} & x_{124} & 0 & -x_{324} & x_{134} & 0 & -x_{334} & 0 & 0 & 0 & -x_{145} & 0 & x_{345} \\
-x_{214} & x_{314} & 0 & -x_{224} & x_{324} & 0 & -x_{234} & x_{334} & 0 & 0 & 0 & 0 & x_{245} & -x_{345} & 0 \\
0 & -x_{115} & x_{215} & 0 & -x_{125} & x_{225} & 0 & -x_{135} & x_{235} & 0 & -x_{145} & x_{245} & 0 & 0 & 0 \\
x_{115} & 0 & -x_{315} & x_{125} & 0 & -x_{325} & x_{135} & 0 & -x_{335} & x_{145} & 0 & -x_{345} & 0 & 0 & 0 \\
-x_{215} & x_{315} & 0 & -x_{225} & x_{325} & 0 & -x_{235} & x_{335} & 0 & -x_{245} & x_{345} & 0 & 0 & 0 & 0 \\
0 & -x_{116} & x_{216} & 0 & -x_{126} & x_{226} & 0 & -x_{136} & x_{236} & 0 & -x_{146} & x_{246} & 0 & -x_{156} & x_{256} \\
x_{116} & 0 & -x_{316} & x_{126} & 0 & -x_{326} & x_{136} & 0 & -x_{336} & x_{146} & 0 & -x_{346} & x_{156} & 0 & -x_{356} \\
-x_{216} & x_{316} & 0 & -x_{226} & x_{326} & 0 & -x_{236} & x_{336} & 0 & -x_{246} & x_{346} & 0 & -x_{256} & x_{356} & 0 \\
0 & -x_{117} & x_{217} & 0 & -x_{127} & x_{227} & 0 & -x_{137} & x_{237} & 0 & -x_{147} & x_{247} & 0 & -x_{157} & x_{257} \\
x_{117} & 0 & -x_{317} & x_{127} & 0 & -x_{327} & x_{137} & 0 & -x_{337} & x_{147} & 0 & -x_{347} & x_{157} & 0 & -x_{357} \\
-x_{217} & x_{317} & 0 & -x_{227} & x_{327} & 0 & -x_{237} & x_{337} & 0 & -x_{247} & x_{347} & 0 & -x_{257} & x_{357} & 0
\end{array}\right.
$$

Bertini team [N. Daleo & J. Hauenstein] to the rescue!

How do we know that the 21×21 determinant $\det \varphi_T$ is irreducible (and hence minimally defines the ideal of $\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1,6))))$?

A computation in Bertini provides missing ingredient:

Proposition*

The hypersurface $\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1,6))) \subset \mathbb{P}^{62}$ has degree 21.

We have an irreducible hypersurface of degree 21 inside of the zero-locus of a degree 21 equation!

Proposition*

The hypersurface $\sigma_8(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1,10))) \subset \mathbb{P}^{164}$ has degree 33.

A similar 33×33 exterior flattening provides the equation.

Proposition*

The hypersurface $\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(2,5))) \subset \mathbb{P}^{59}$ has degree 6.

No 6×6 flattening... So now what?

Find equations in the ideal

Consider $\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(2,5))) \subset \mathbb{P}^{59}$. Look for equations in $R = \mathbb{C}[x_0, \ldots, x_{59}]$. The Hilbert function of R starts like this:

$$\begin{array}{rcrrrrrrl} d & = & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ HF_R(d) & = & 60 & 1830 & 37820 & 595665 & 7624512 & 82598880 & \ldots \\ HF_{R/I}(d) & = & 0 & 0 & 0 & 0 & 0 & 82598879 & \ldots \\ \dim(I_d) & = & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \end{array}$$

To find the space of linear forms in the ideal, compute:

```
M = matrix apply(60,i-> {makeRank5(2,2,5)} );
rank M
```

Naively, to find the space of sextics in the ideal: compute 82598880 points on the variety, evaluate them on the 82598880 monomials of degree 6 and compute the kernel of the resulting 82598880×82598880 matrix.

Symmetry

- The symmetry group of

$$\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(2,5))) \subset \mathbb{P}^{59} = \mathbb{P}\left(A \otimes \bigwedge^2 B\right)$$

  is change of coordinates in each factor,

$$GL(A) \times GL(B)$$

- A large group acts so we can use tools from Representation Theory!
- This symmetry is a powerful tool and we *should* exploit it!

Find equations in the ideal using representation theory

- Module notation: $S^d\left(A\otimes\bigwedge^2 B\right) = \mathbb{C}[p_{ijk} \mid p_{ijk} = -p_{ikj}]_d$.

- Fact: $S^d\left(A\otimes\bigwedge^2 B\right)$is a $GL(A)\times GL(B)$-module.
  The coordinate ring $\mathbb{C}[A\otimes\bigwedge^2 B]$ has an isotypic decomposition:

$$\begin{array}{ccc} \bigoplus_d \mathbb{C}[A\otimes\bigwedge^2 B]_d & = & \bigoplus_d \left(\bigoplus_{\lambda\vdash d,\pi\vdash 2d} (S_\lambda A\otimes S_\pi B)\otimes\mathbb{C}^{m_{\lambda,\pi}}\right) \\ \cup & & \cup \\ \bigoplus_d \mathcal{I}_d(X) & = & \left(\bigoplus_{\lambda\vdash d,\pi\vdash d} \bigoplus_d \mathcal{I}_d(X)_{\lambda,\pi}\right) \end{array}$$

  - $S_\lambda A\otimes S_\pi B$: Schur modules indexed by partitions λ and π,
  - $\mathbb{C}^{m_{\lambda,\pi}}$: multiplicity space.

- Given λ,π and the multiplicity $m_{\lambda,\pi}$, there is a combinatorial algorithm for constructing polynomials!

(Modified version of Landsberg-Manivel'04 algorithm)

Invariants via Young Symmetrizers

Use LiE: `sym_tensor(6,[1,0]^[0,0,1,0,0],A2A5)`
$\Rightarrow$ only one $\mathrm{SL}(3) \times \mathrm{SL}(6)$ invariant of degree 6 in $\mathbb{C}[A \otimes \bigwedge^2 B]$.

Start with partitions $(2, 2, 2)$ and $(3, 3, 3, 3, 3, 3)$ associated (respectively) to the trivial representations of $\mathrm{GL}(3)$ and $\mathrm{GL}(6)$ in degrees 6 and 18 respectively.

Find fillings:

$$\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & f \\ \hline \end{array} \quad \otimes \quad \begin{array}{|c|c|c|}\hline a & b & c \\ \hline a & b & d \\ \hline a & d & e \\ \hline b & d & f \\ \hline c & e & f \\ \hline c & e & f \\ \hline \end{array}$$

Use fillings as input to the Young Symmetrizer algorithm.

Invariants via Young Symmetrizers

Construct a generic polynomial (in auxiliary variables):

1. To the filling $\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & f \\ \hline \end{array}$ associate $p_V = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \cdot \begin{vmatrix} c_1 & c_2 & c_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \end{vmatrix}$

2. To the filling $\begin{array}{|c|c|c|}\hline a & b & c \\ \hline a & b & d \\ \hline a & d & e \\ \hline b & d & f \\ \hline c & e & f \\ \hline c & e & f \\ \hline \end{array}$ associate $p_W =$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} & d_{26} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} & d_{36} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{vmatrix} \cdot \begin{vmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ d_{11} & d_{12} & d_{13} & d_{14} & d_{15} & d_{16} \\ e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \end{vmatrix}.$$

3. Extract the terms $p_V p_W$: replace $a_i \cdot (a_{1j} \wedge a_{2k} \wedge a_{3l})$ with $x_{i,j,k,l}$
4. Repeat previous step for $b, c, \ldots, f$.

The resulting polynomial will be a polynomial in the image of the Young Symmetrizer associated to our initial fillings.

Solution to Problem 6.5 in [Abo-Wan 2012]

Theorem (Abo-Daleo-Hauenstein-O.)

The hypersurface $\sigma_5(\mathbb{P}^2 \times \mathbb{G}(2,5)) \subset \mathbb{P}^{59}$ is defined by the image of the Young symmetrizer produced by the recipe given by the filling

$$\begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d & f \\ \hline \end{array} \quad \otimes \quad \begin{array}{|c|c|c|}\hline a & b & c \\ \hline a & b & d \\ \hline a & d & e \\ \hline b & d & f \\ \hline c & e & f \\ \hline c & e & f \\ \hline \end{array}.$$

The resulting polynomial has precisely 10080 monomials, 5040 of which have coefficient $+1$ and 5040 of which have coefficient -1.
Download from ancillary files associated to the arXiv version of our paper!

"Theorem" not "Theorem*" because there is no lower degree invariant equation.

An Ottaviani-type expression?

We suppose that this equation may have an expression as a root of a determinant of a special matrix, similar to Ottaviani's degree 15 equation in [Ottaviani'09], however our initial attempts at finding such an expression were unsuccessful.

A natural guess is to start from $T \in A \otimes \wedge^3 B$ and produce the 18×36 matrix

$$A_T \colon (B \otimes B)^* \to (A \otimes B),$$

which as rank 3 when T has rank 1 and rank $\leq 3r$ when T has rank r.

However, this map actually factors through a map

$$\wedge^2 B^* \to (A \otimes B)$$

but this matrix is 18×15, and it's maximum rank is 15. This means that this construction cannot distinguish rank 5 tensors from rank 6 tensors.

Numerical Algebraic Geometry: Bertini

Let $\mathcal{H}$ be an irreducible hypersurface and $\mathcal{L}$ be a line so that $\deg \mathcal{H} = |\mathcal{H} \cap \mathcal{L}|$.

1. Generate a point $x \in \mathcal{H} \cap \mathcal{L}$. Initialize $\mathcal{W} := \{x\}$.
2. Perform a random monodromy loop starting at the points in $\mathcal{W}$:
   (a) Pick a random loop $\mathcal{M}(t)$ in the space of lines so that $\mathcal{M}(0) = \mathcal{M}(1) = \mathcal{L}$.
   (b) Trace the curves $\mathcal{H} \cap \mathcal{M}(t)$ starting at the points in $\mathcal{W}$ at $t = 0$ to compute the endpoints $\mathcal{E}$ at $t = 1$. (Hence, $E \subset \mathcal{H} \cap \mathcal{L}$).
   (c) Update $\mathcal{W} := \mathcal{W} \cup \mathcal{E}$.
3. Repeat (2) until the trace test verifies that $\mathcal{W} = \mathcal{H} \cap \mathcal{L}$.

Proposition*

The hypersurface $\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(2,5))) \subset \mathbb{P}^{59}$ *has degree 6.*

In our execution of the procedure for the hypersurface $\mathcal{H} = \sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(2,5)))$, it took 6 random monodromy loops to compute the six points in $\mathcal{H} \cap \mathcal{L}$. The total procedure lasted 50 seconds using a single 2.3 GHz core of an AMD Opteron 6376 processor.

Proposition*

The hypersurface $\sigma_5(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1,6))) \subset \mathbb{P}^{62}$ *has degree 21.*

Proposition*

The hypersurface $\sigma_8(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{G}(1,10))) \subset \mathbb{P}^{164}$ *has degree 33.*

In our execution, it took 13 and 12 random monodromy loops to yield the degree many points for these cases, respectively. Using a total of sixteen 2.3 GHz cores, the total procedure lasted 2.5 and 32 minutes, respectively.