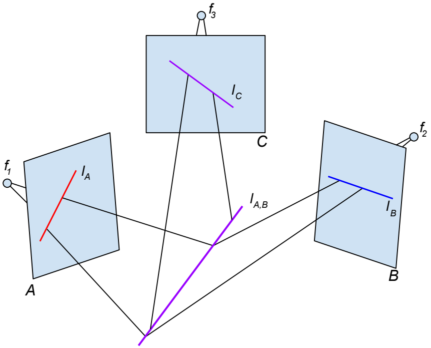
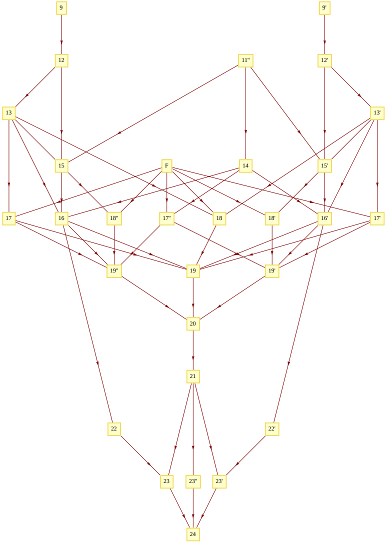
# The trifocal ideal




November 27, 2017

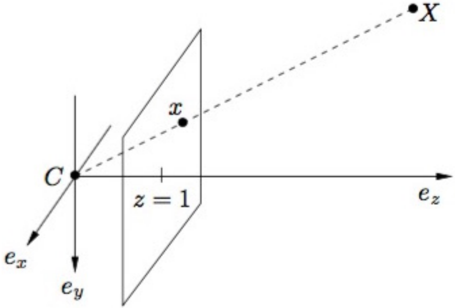


# Image reconstruction

Google "Duomo Florence"

Try to reconstruct a 3D model of this beautiful structure from 2D images.

# The Pinhole Camera




Model the 3D world as projective 3-space, $\mathbb{P}^3$.
Homogeneous coordinates $[X] := [X_0 : X_1 : X_2 : X_3]$.

The 2D image is modeled by projective 2-space, $\mathbb{P}^2$.
Homogeneous coordinates $[x] := [x_0 : x_1 : x_2]$.

The standard pinhole camera is modeled by projection $\mathbb{P}^3 \to \mathbb{P}^2$.

The projection is induced from a linear map on affine spaces, represented by a $3 \times 4$ *camera matrix* $A$.

The projection is simply $[X] \mapsto [A.X] = [x]$

Assume $A$ has full rank. Can choose coordinates so that $A = \left(I_3 | \vec{a}_4\right)$,
where $\vec{a}_4$ will be the coordinates of the image of the focal point of the camera.

# Fundamental Matrices

Point correspondences between two images (cameras $A_1$ and $A_2$) are encoded in the $3 \times 3$ *fundamental matrix*, $\mathtt{F}$, defined by the algebraic conditions:

$$\text{If } A_1 X = x \text{ and } A_2 X = x' \text{ (a point-point correspondence) then } x^\top \mathtt{F} x' = 0.$$

Given the stacked camera matrix is (after change in coordinates)

$$M = (A_1^\top \mid A_2^\top) \cong \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ x_{1,1} & x_{1,2} & x_{1,3} & x_{2,1} & x_{2,2} & x_{2,3} \end{array}\right),$$

the entries of $\mathtt{F}$ are the $4 \times 4$ minors of $M$ using 2 columns from each block:

$$F(M) = \begin{pmatrix} 0 & x_{1,3} - x_{2,3} & -x_{1,2} + x_{2,2} \\ -x_{1,3} + x_{2,3} & 0 & x_{1,1} - x_{2,1} \\ x_{1,2} - x_{2,2} & -x_{1,1} + x_{2,1} & 0 \end{pmatrix}.$$

# Fundamental Matrices

**Theorem (Hartley-Zisserman Thm. 9.10)**

*The fundamental matrix determines the camera matrices up to projective transformation. That is, if $(P, P')$ and $(\tilde{P}, \tilde{P}')$ are two pairs of matrices with the same fundamental matrix $\mathtt{F}$, then there exists a nonsingular $4 \times 4$ matrix $H$ such that $P = \tilde{P}H$ and $P' = \tilde{P}'H$.*

If the camera matrices are not known, $\mathtt{F}$ has 9 homogeneous parameters (only defined up to scale) and must have rank 2, so it is determined by the linear conditions imposed by 7 point-point correspondences. Camera matrices can be reconstructed (up to a projective change of coordinates and some finite ambiguities).

Once the two camera matrices are reconstructed, triangulation allows us to reconstruct the 3D world points associated to each point-point correspondence.

Random Sample Consensus (RANSAC) can be used to find a fundamental matrix and cameras that best explain the point correspondences.

Typical issue: sometimes difficult to determine 7 inliers in two different images.

Possible improvement: use more images to reduce the number of required inliers.

# Multiview Geometry

[Hartley-Zisserman]

A *camera* is a projection

$$\begin{array}{rcl} A\colon \mathbb{P}^3 & \dashrightarrow & \mathbb{P}^2 \\ x & \mapsto & Ax, \end{array}$$

given by a $3 \times 4$ camera matrix $A$.

For *fixed* $(A_1, \dots A_n)$ cameras in general position get a map

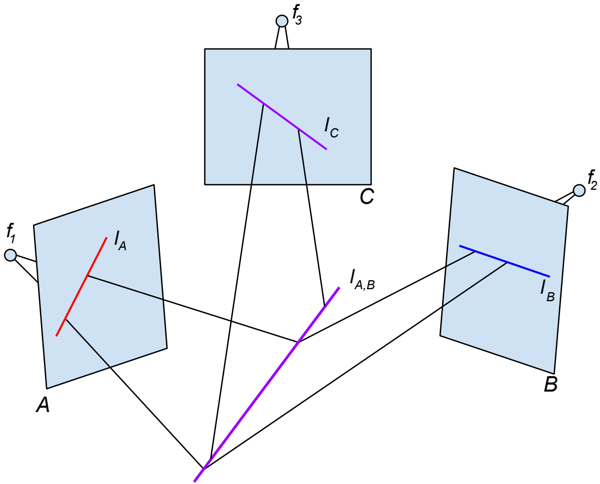
$$\begin{array}{rcl} \phi\colon \mathbb{P}^3 & \dashrightarrow & (\mathbb{P}^2)^{\times n} \\ x & \mapsto & (A_1 x, A_2 x, \dots, A_n x) \end{array}$$

parametrizing a multiview variety.

[AST *(A Hilbert scheme in Computer Vision)*] found the prime ideal for generic $(A_1, \dots, A_n)$ and identified it in a distinguished component of the multi-graded Hilbert scheme.

But suppose we didn't know the positions of the cameras, or even if the points arose this way...

# From geometry to bilinear maps and tensors




line: $l_A \subset A$
focal point: $f_1$
plane: $\{l_A, f_1\}$

line: $l_B \subset B$
focal point: $f_2$
plane: $\{l_B, f_2\}$

intersect:
$\{l_A, f_1\} \cap \{l_B, f_2\} = l_{A,B}$.
$\{l_{A,B}, f_3\} \cap C =: l_C$

We've constructed a bi-linear map: $A \times B \to C$ *i.e.* a (trifocal) tensor in $A^* \otimes B^* \otimes C$.

# Multiview Geometry to Trifocal Tensors

Set $n = 3$ cameras.
The maximal minors of the $4 \times 9$ matrix $(A_1^t \mid A_2^t \mid A_3^t)$ give coordinates of a point on the Grassmannian $Gr(4, 9)$.

A trifocal tensor is a point whose coordinates are the $4 \times 4$ minors of $(A_1^t \mid A_2^t \mid A_3^t)$ using one column from the first two matrices and two from the third.

The trifocal variety can be constructed as a projection of the Grassmannian.

The camera matrices $A_i$ are only defined up to scale, and the trifocal variety is birational to the GIT quotient $\widehat{Gr(4,9)}//(\mathbb{C}^*)^3$.

# Multiview Geometry to Trifocal Tensors

Set $n = 3$ cameras, with matrices $A_1, A_2, A_3$.
The maximal minors of the $4 \times 9$ matrix $(A_1^t \mid A_2^t \mid A_3^t)$ give coordinates of a point on the Grassmannian $Gr(4,9) \subset \mathbb{P}(\bigwedge^4 \mathbb{C}^9)$.

Let $A^*, B^*, C^* \cong \mathbb{C}^3$ be the column spaces of $A_1^t, A_2^t, A_3^t$ respectively.

Have a $G := \mathrm{SL}(A) \times \mathrm{SL}(B) \times \mathrm{SL}(C)$-invariant subspace

$$\bigwedge^4 \mathbb{C}^9 = \bigwedge^4 (A^* \oplus B^* \oplus C^*) \supset A^* \otimes B^* \otimes \bigwedge^2 C^* \cong A^* \otimes B^* \otimes C$$

–the maximal minors that use 1 column from each of the first two blocks and 2 from the third.

The $G$-equivariant projection $Gr(4,9) \dashrightarrow \mathbb{P}(A^* \otimes B^* \otimes C)$ defines the trifocal tensors (automatically $G$-invariant).

The fibers of the projection are scaling in each block matrix, so the image $X$ (the trifocal variety) is an 18-dimensional $G$-variety in $\mathbb{P}(A^* \otimes B^* \otimes C)$.

# General multi-focal tensors

Stack camera matrices $A_i\colon \mathbb{C}^N \to \mathbb{C}^n$ to form the $N \times m \cdot n$ matrix

$$M = \left( \ A_1^\top \ \middle| \ A_2^\top \ \middle| \ \dots \ \middle| \ A_m^\top \ \right).$$

$M$ has maximal rank on an open set of camera configurations.

Fix a partition $\pi \vdash N$, with $\#\pi \leq m$ parts, $N \leq m \cdot n$, and $\pi_1 \leq n$:

$Z_\pi$ is the variety of the maximal minors of $M$ using $\pi_i$ columns from $A_i^\top$.

- $Z_\pi$ is an equivariant projection from the Grassmannian $Gr(N, mn)$.
- $Z_\pi$ is invariant under the action of $\mathrm{GL}(n)^{\times \#\pi}$.
  - see also [Hartley-Zisserman], [Aholt-Sturmfels-Thomas], [Heyden], [Triggs], [Faugeras-Mourrain]...

We are most interested in the case $N = 4$, $m = 4$ $n = 3$:

$Z_{2,2}$ is the variety of *fundamental matrices* (skew-symmetric $3 \times 3$ matrices).

$Z_{2,1,1}$ is the variety of *trifocal tensors* (special $3 \times 3 \times 3$ tensors). (see [Aholt-O.'14]).

$Z_{1,1,1,1}$ is the variety of *quadrifocal tensors* (special $3 \times 3 \times 3 \times 3$ tensors).

# Main Theorem

**Theorem (Aholt–O. (2014))**

*Let $X$ denote the trifocal variety. The prime ideal $I(X)$ is minimally generated by 10 polynomials in degree 3, 81 polynomials in degree 5, and 1980 polynomials in degree 6.*

- The polynomials have small integer coefficients ($|\cdot| < 5$) and up to 732 monomials each.
- These are the lowest degree generators of the ideal of the trifocal variety.
- Improves [Alzati–Tortora'10]'s set-theoretic equations (and lowers degrees).
- Gives a test to determine whether $T \in A^* \otimes B^* \otimes C$ is a trifocal tensor.

# Proof ingredients

In general, implicitization – finding the ideal in 27 variables of an 18-dimensional variety – is difficult. Brute force is unlikely to work.

Here's an outline of how we attacked the problem:

- Representation theory of $G$-modules of polynomials
- Numerical Algebraic Geometry
- Invariant Theory & Classification of Normal Forms
- Commutative Algebra

# Find polynomials by any means necessary

Naively compute the Hilbert function for $I(X) \subset R = \mathbb{C}[x_1, \ldots, x_{27}]$ for as high a degree as possible:

$$
\begin{aligned}
HF_R &= (1, 27, 378, 3654, 27405, 169911, 906192, 4272048, 18156204, 70607460, \ldots \\
HF_{R/I} &= (1, 27, 378, 3644, 27135, \\
\dim_I &= (0, \quad 0, \quad 0, \quad 10, \quad 270,
\end{aligned}
$$

- Set $N = \dim R_d$. Make an $N \times N$ matrix of $N$ monomials of degree $d$ evaluated at $N$ random points of $X$. Compute its kernel.

# Find polynomials by any means necessary

Naively compute the Hilbert function for $I(X) \subset R = \mathbb{C}[x_1, \ldots, x_{27}]$ for as high a degree as possible using Representation Theory:

$$HF_R = (1, 27, 378, 3654, 27405, 169911, 906192, 4272048, 18156204, 70607460, \ldots$$
$$HF_{R/I} = (1, 27, 378, 3644, 27135, 166050, 865860, 3942162, 15966072, 58409126, ?$$
$$\dim_I = (0, \ 0, \ 0, \ 10, \ 270, \ 3861, \ 40332, \ 329886, \ 2190132, 12198334, ?$$

- Set $N = \dim R_d$. Make an $N \times N$ matrix of $N$ monomials of degree $d$ evaluated at $N$ random points of $X$. Compute its kernel.
- Use Representation Theory of tensor products of Schur modules to aid computations:
  - $G$-structure reduces the computation to computing kernels of several smaller blocks.
  - We also get a $G$-module description of the lowest degree parts of $I(X)$ and $\mathbb{C}[X]$.

# Representation theory results

We applied machinery introduced by Landsberg and Manivel to perform our representation theoretic computations.

Starting in degree 3, we computed the highest weight spaces of $G$-modules of polynomials spanning the ideal $I(X)$ in each degree up to 9.

Using Macaulay2[1], we found the minimal generators in those degrees.

$$
\begin{aligned}
HF_R &= (1, 27, 378, 3654, 27405, 169911, 906192, 4272048, 18156204, 70607460, \ldots \\
HF_{R/I} &= (1, 27, 378, 3644, 27135, 166050, 865860, 3942162, 15966072, 58409126, ? \\
\dim_I &= (0, \ 0, \ 0, \ 10, \ 270, \ 3861, \ 40332, \ 329886, \ 2190132, 12198334, ? \\
{}_{mingens\ I} &= (0, \ 0, \ 0, \ 10, \ 0, \ 81, \ 1980, \ 0, \ 0, \ 0, ?
\end{aligned}
$$

[1]Grayson-Stillman

# Main Theorem (Invariant description)

Let $X \subset \mathbb{P}(A^* \otimes B^* \otimes C)$ be the trifocal variety, $G = \mathrm{SL}(A) \times \mathrm{SL}(B) \times \mathrm{SL}(C)$.
Given partitions $\lambda, \mu, \nu$ of $d$, let $S_\lambda S_\mu S_\nu$ denote the $G$-module $S_\lambda A \otimes S_\mu B \otimes S_\nu C^*$ of polynomials in $S^d(A \otimes B \otimes C^*)$.

## Theorem (Aholt–O.)

*The ideal of $X$ is generated by the following $G$-modules.*

$M_3 = \bigwedge^3 \bigwedge^3 S^3,$ *10 cubics,*

$M_5 = S_{221} S_{221} S_{311} \oplus S_{221} S_{221} S_{221},$ *81 quintics,*

$M_6 = S_{222} S_{33} S_{33} \oplus S_{33} S_{222} S_{33} \oplus S_{222} S_{33} S_{411} \oplus S_{33} S_{222} S_{411}$
$\oplus S_{33} S_{411} S_{222} \oplus S_{411} S_{33} S_{222} \oplus S_{33} S_{33} S_{222}$
$\oplus S_{33} S_{321} S_{321} \oplus S_{321} S_{33} S_{321},$ *and 1980 sextics.*

# Normal form of a trifocal tensor (main orbit)

A (rank 4) tensor in $A^* \otimes B^* \otimes C$:

$$T = a_1 \otimes b_2 \otimes c_1 + a_3 \otimes b_1 \otimes c_1 + a_2 \otimes b_2 \otimes c_2 + a_3 \otimes b_3 \otimes c_3.$$

3 contractions:

$$T(A) = \begin{pmatrix} a_3 & 0 & 0 \\ a_1 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix},$$

P-Rank$_A = 3$

$$T(B) = \begin{pmatrix} b_2 & 0 & 0 \\ 0 & b_2 & 0 \\ b_1 & 0 & b_3 \end{pmatrix},$$

P-Rank$_B = 3$

$$T(C) = \begin{pmatrix} 0 & c_1 & 0 \\ 0 & c_2 & 0 \\ c_1 & 0 & c_3 \end{pmatrix}$$

P-Rank$_C = 2$

3 flattenings to $3 \times 9$ matries:

| $bc$ | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $a_1$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $a_2$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $a_3$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

,

F-Rank$_A = 3$

| $ac$ | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $b_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $b_2$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $b_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

,

F-Rank$_B = 3$

| $ab$ | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $c_1$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $c_2$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $c_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

,

F-Rank$_C = 3$

# Evgeniy Martyushev's Normal form ("F" on Nurmiev's list)

A (rank 4) tensor in $A^* \otimes B^* \otimes C$:

$$E = a_1 \otimes (b_2 + b_3) \otimes c_1 + (-a_2 - a_3) \otimes b_1 \otimes c_1 + a_2 \otimes (b_3 - b_1) \otimes c_2 + (a_1 - a_3) \otimes b_2 \otimes c_2$$

3 contractions:

$$E(A)\begin{pmatrix} -a_2 - a_3 & -a_2 & 0 \\ a_1 & a_1 - a_3 & 0 \\ a_1 & a_2 & 0 \end{pmatrix}$$

P-Rank$_A = 2$

$$E(B)\begin{pmatrix} b_2 + b_3 & b_2 & 0 \\ -b_1 & -b_1 + b_3 & 0 \\ -b_1 & -b_2 & 0 \end{pmatrix}$$

P-Rank$_B = 2$

$$E(C)\begin{pmatrix} 0 & c_1 + c_2 & c_1 \\ -c_1 - c_2 & 0 & c_2 \\ -c_1 & -c_2 & 0 \end{pmatrix}$$

P-Rank$_C = 2$

3 flattenings to $3 \times 9$ matries:

| $bc$ | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $a_1$ | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| $a_2$ | −1 | −1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $a_3$ | −1 | 0 | 0 | 0 | −1 | 0 | 0 | 0 | 0 |

F-Rank$_A = 3$

| $ac$ | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $b_1$ | 0 | 0 | 0 | −1 | −1 | 0 | −1 | 0 | 0 |
| $b_2$ | 1 | 1 | 0 | 0 | 0 | 0 | 0 | −1 | 0 |
| $b_3$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

F-Rank$_B = 3$

| $ab$ | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $c_1$ | 0 | 1 | 1 | −1 | 0 | 0 | −1 | 0 | 0 |
| $c_2$ | 0 | 1 | 0 | −1 | 0 | 1 | 0 | −1 | 0 |
| $c_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

F-Rank$_C = 2$

# Normal form "9"

A (rank 3) tensor in $A^* \otimes B^* \otimes C$:

$$S = a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_2 \otimes c_2 + a_2 \otimes b_2 \otimes c_2 + a_2 \otimes b_3 \otimes c_3$$

3 contractions:

$$S(A) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1 + a_2 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$$

$\text{P-Rank}_A = 3$

$$S(B) = \begin{pmatrix} b_1 & b_2 & 0 \\ 0 & b_2 & b_3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{P-Rank}_B = 2$

$$S(C) = \begin{pmatrix} c_1 & c_2 & 0 \\ 0 & c_2 & c_3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{P-Rank}_C = 2$

3 flattenings to $3 \times 9$ matries:

| bc | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $a_1$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $a_2$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $a_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

,

$\text{F-Rank}_A = 2$

| ac | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $b_1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b_2$ | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $b_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |

,

$\text{F-Rank}_B = 3$

| ab | 11 | 12 | 13 | 21 | 22 | 23 | 31 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|---|
| $c_1$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $c_2$ | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $c_3$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |

,

$\text{F-Rank}_C = 3$

# Polarization: how to write down 10 special cubics

Think of a $3 \times 3 \times 3$ tensor as three $3 \times 3$ blocks of a matrix $T = (T_1 \mid T_2 \mid T_3)$

$$= \left(\begin{array}{ccc|ccc|ccc} T000 & T100 & T200 & T001 & T101 & T201 & T002 & T102 & T202 \\ T010 & T110 & T210 & T011 & T111 & T211 & T012 & T112 & T212 \\ T020 & T120 & T220 & T021 & T121 & T221 & T022 & T122 & T222 \end{array}\right)$$

Use dummy variables $z1, z2, z3$ to form the generic contraction:

$$T(z) = z1\,T_1 + z2\,T_2 + z3\,T_3.$$

$$\begin{pmatrix} T000z0 + T001z1 + T002z2 & T100z0 + T101z1 + T102z2 & T200z0 + T201z1 + T202z2 \\ T010z0 + T011z1 + T012z2 & T110z0 + T111z1 + T112z2 & T210z0 + T211z1 + T212z2 \\ T020z0 + T021z1 + T022z2 & T120z0 + T121z1 + T122z2 & T220z0 + T221z1 + T222z2 \end{pmatrix}$$

If $T$ is a trifocal tensor then $T(z)$ is a bifocal tensor for all $z$.

Recall $F$ is a bifocal tensor if and only if it has rank 2.

Therefore $\det(T(z)) \equiv 0$. The coefficients in $z$ are 10 cubic equations in the entries of $T$.

Gives a basis of the Schur Module $(S^3 V_1 \otimes \bigwedge^3 V_2 \otimes \bigwedge^3 V_3)^*$.

# Bertini and the degree

Bertini[2] is a system for *Numerical Algebraic Geometry*. Computes a numerical primary decomposition, *ignoring embedded components and multiplicities*.
6.5hours (2 processors) or 10minutes (48 processors – Jon's cluster):

**Computation (Hauenstein/Bertini)**

*Let $M_3$ denote the 10 coefficients (in $x_1, x_2, x_3$) of the cubic $\det(x_1 T_1 + x_2 T_2 + x_3 T_3)$. The zero set of $M_3$ has precisely 4 components:*
*In codimension 7 there are 2 components, each of degree 36.*
*In codimension 8 there is 1 component of degree 297.*
*In codimension 10 there is 1 component of degree 1035.*

Components are F-Rank-$(2,3,3)$, F-Rank-$(3,2,3)$, Trifocal, P-Rank-$(2,2,2)$.
Corroborates result using Nurmiev's poset (next).
We also learn the degree of each integral variety!

[2] [Bates, Hauenstein, Sommese, Wampler]

# Orbit classification and normal forms

$SL(3)^{\times 3}$ acts on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ with infinitely many orbits, but they are classified[1].

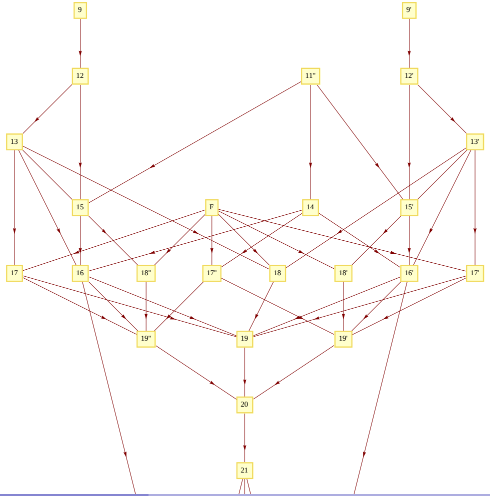
Get normal forms for every orbit.
Easy to check which orbits are in the zero-locus of a set of polynomials.

The zero-locus of the polynomials coming from P-Rank-$(3, 3, 2)$ consist of the following orbits with poset structure (Nurmiev)

4 sources in the poset means that P-Rank-$(3, 3, 2)$ is the union of 4 orbit closures (irreducible varieties).

[[1] Thrall-Chanler'38, Ng '95, Vinberg-Èlašvili'78, Nurmiev'00]

Nurmiev Poset:



# Determine whether a given tensor is trifocal

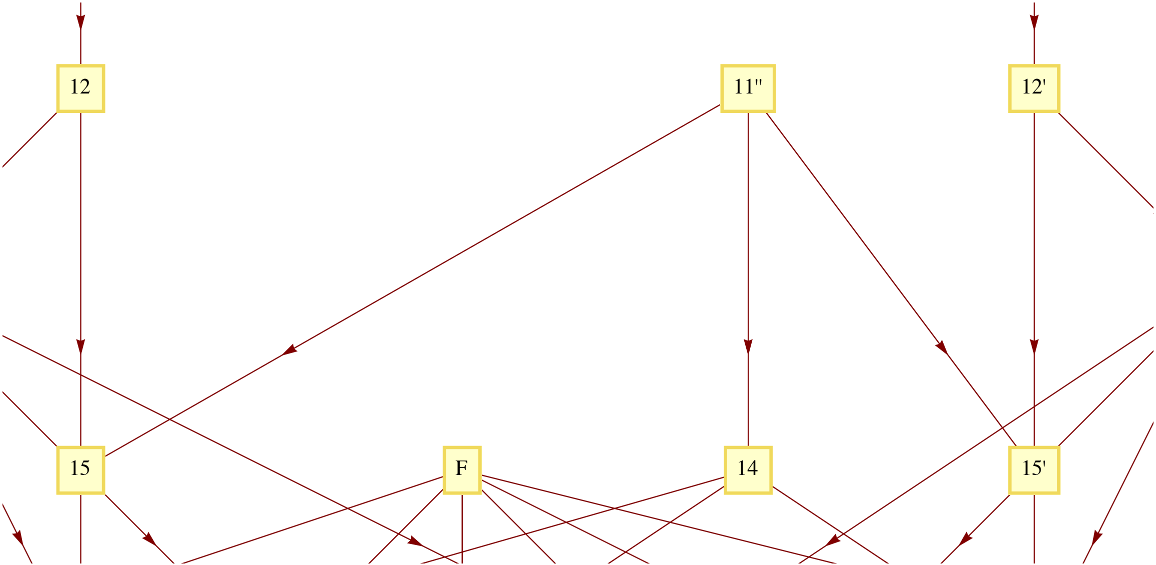
- Separate orbits:
  - The conditions P-Rank-$(3, 3, 2)$ and F-Rank-$(3, 3, 3)$ (vanishing and non-vanishing) separate the trifocal orbit from the other orbits.
- Up to generic choices (generically complete set of equations):
  - [AT'10] showed 3 cubic, 3 quintic and 2 sextic equations, and 2 degree 4 inequalities define a Zariski open subset of trifocal tensors.
- Set-theoretic defining equations
  - [AT'10] used 10 degree 3, 20 degree 9, 6 degree 12 equations.
  - Add in more polynomials to eliminate other geometric components from P-Rank-$(3, 3, 2)$.
- Ideal-theoretic defining equations
  - Add in more polynomials to eliminate other embedded components from the ideal.

# Notions of degree

[see Sturmfels-Trung-Vogel'95]

- Bertini finds 297 simple points in a random codimension-sized linear space intersecting the trifocal variety, so the degree of the trifocal variety is 297 (...up to numerical precision and our trust in Bertini).
- A Gröbner basis computation in Macaulay2 shows that $J = \langle M_3 + M_5 + M_6 \rangle$, (the 10+81+1980 polynomials) has degree 297 [the leading coefficient of the Hilbert polynomial and the degree of the top dimensional component].
- Conclude that the top dimensional component of $J$ agrees with the trifocal variety $X$, and the multiplicity of $I(X)$ is 1 in $J$.
- The set-theoretic result implies that the *geometric degree* of $J$ (sum of degrees of all isolate components) is 297.
- What about possible embedded components? i.e. can we show that the *arithmetic degree* (sum of degrees of all embedded and isolated components) of $J$ is 297?

# Zero divisors and Hilbert series



# Zero divisors and Hilbert series

The trifocal variety corresponds to the closure of orbit 11" from Nurmiev. The first orbits in the closure are 14, 15, 15'.

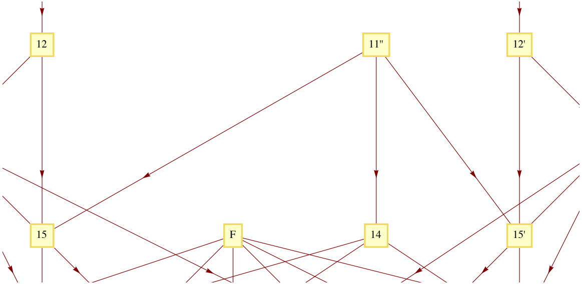
$G$ is connected. So the associated primes in a primary decomposition are $G$-invariant.

Possible minimal primary decomposition of $J$:

$$J = I(X) \cap Q_{14} \cap Q_{15} \cap Q_{15'},$$

with primary ideals $Q_i$ and assoc. primes $P_i$.

The orbit closure poset is also a poset of $G$-stable primes.




We want to show that the primary components $Q_{14}, Q_{15}$, and $Q_{15'}$ do not occur.

# Zero divisors and Hilbert series[4]

Basic fact[3]: The set of zero divisors of $R/J$ is the union of prime ideals that contain $J$.
Suppose $J = I(X) \cap Q_{14} \cap Q_{15} \cap Q_{15'}$, $ass(Q_i) = P_i$.
Suppose $f \in P_i$ has degree $d$. If $f$ is not a zero-divisor of $R/J$, the following sequence is exact

$$0 \longrightarrow (R/J)(-d) \xrightarrow{f} R/J \longrightarrow R/(J+f) \longrightarrow 0.$$

Since $H_{(R/J)(-d)}(t) = t^d H_{(R/J)}(t)$, by the additivity of Hilbert series,

$$(1 - t^d) H_{R/J}(t) = H_{R/(J+f)}(t). \tag{1}$$

If $f$ is a zero-divisor, (**??**) fails for some $t$.
Else $f$ is not a zero-divisor and $J \not\subset P_i$.

[3] Atiyah-MacDonald Prop. 4.7

[4] Thanks to Steven Sam for suggesting we try this computation.

# Zero divisors and Hilbert series

- We selected $f$ in $P_{14}$, (resp. $P_{15}$, $P_{15'}$), and computed a Gröbner basis and the Hilbert series of $J+f$ in 10 hours (resp. 45hrs) on a server with 8 processors and 16 GB of RAM.
- For each $f$ we confirmed the validity of

$$(1-t^d)H_{R/J}(t) = H_{R/(J+f)}(t).$$

- So, $P_{14}$, $P_{15}$, $P_{15'}$ are not imbedded in $J$.
- Therefore $J$ is prime and agrees with $I(X)$.

# Summary

The trifocal variety and its ideal have been studied using the following tools

- Symmetry of the variety and the Representation Theory of its ideal
  – found all polynomials in low degree (up to 9)
- Numerical Algebraic Geometry (Bertini) – computing numerical primary decomposition of the first polynomials we encountered
  – found the degree of the trifocal variety
- Classification of orbits - a list of all possible $G$-varieties and minimal $G$-stable prime ideals
- Symbolic (Gröbner basis) computations – degree, Hilbert Series...
- Commutative algebra – ruling out possible embedded components of the known ideal

# Summary

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THANKS!

We would like to give a complete algebraic description of the quadrifocal variety $Z$ by finding the generators of its defining ideal $I(Z)$ (the implicit defining equations of the model).

Naively, to find out if the variety $Z$ lives in a linear subspace, put the coordinates of 81 points in the rows of a matrix $P = \begin{pmatrix} —Z_1— \\ \vdots \\ —Z_{81}— \end{pmatrix}$ The null space of $P$ is the vector space of linear forms vanishing on the 81 points (and very likely all of $Z$).

If $\mathrm{Null}(P) = 0$, there are no linear forms in the ideal of $Z$.

In higher degree $d$ we can Veronese re-embed the points and solve another linear algebra problem to find the space of degree $d$ polynomials vanishing on $Z$.

But the dimensions grow quickly:
$$\begin{pmatrix} 1 & 81 & 3321 & 91881 & 1929501 & 32801517 & 470155077 & 5843355957 & 64276915527 & \ldots \end{pmatrix}$$

The ideal $I(Z)$ is a $G = \mathfrak{S}_4 \ltimes \mathrm{GL}(3)^{\times 4}$-submodule of $R = \mathbb{C}[\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3]$.
$R$ has a $G$-isotypic decomposition:

$$R = \bigoplus_{d \geq 0} \bigoplus_{\pi \vdash_4 d} S_\pi V \otimes M_\pi .$$

where $\pi$ is a multi-partition, the sum is over non-redundant permutations,

- Schur module: $S_\pi V = \bigoplus_{\sigma \in \mathfrak{S}_4/\sim} (S_{\pi_{\sigma.1}} V_1 \otimes S_{\pi_{\sigma.2}} V_2 \otimes S_{\pi_{\sigma.3}} V_3 \otimes S_{\pi_{\sigma.4}} V_4)$
- Multiplicity space (Specht Module): $M_\pi$.

Our tasks for small degree $d$ are the following:

- Compute a basis of $M_\pi$ for each $\pi$.
- Evaluate the highest-weight space of $S_\pi V \otimes M_\pi$ on points of $Z$.
- Obtain a list of $G$-modules (with multiplicity) in $I(Z)$.
- Determine which modules are among the minimal generators.
- Determine the maximal degree of minimal generators.

# Invariant Theory and Young Symmetrizers

Multi-partition: (221, 221, 221,221), Filling: $F = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array}$

Auxiliary polynomial:

$$p = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^1 & c_2^1 & c_3^1 \end{vmatrix} \begin{vmatrix} d_1^1 & d_2^1 \\ e_1^1 & e_2^1 \end{vmatrix} \cdot \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix} \begin{vmatrix} d_1^2 & d_2^2 \\ e_1^2 & e_2^2 \end{vmatrix} \cdot \begin{vmatrix} a_1^3 & a_2^3 & a_3^3 \\ b_1^3 & b_2^3 & b_3^3 \\ c_1^3 & c_2^3 & c_3^3 \end{vmatrix} \begin{vmatrix} d_1^3 & d_2^3 \\ e_1^3 & e_2^3 \end{vmatrix} \cdot \begin{vmatrix} a_1^4 & a_2^4 & a_3^4 \\ b_1^4 & b_2^4 & b_3^4 \\ c_1^4 & c_2^4 & c_3^4 \end{vmatrix} \begin{vmatrix} d_1^4 & d_2^4 \\ e_1^4 & e_2^4 \end{vmatrix}$$

1. Start with $p(a, b, c, d, e, x)$ of multi-degree $(4, 4, 4, 4, 4, 0)$.
2. Replace every $a_i^1 a_j^2 a_k^3 a_l^4$ with $x_{i,j,k,l}$
   - Produce a polynomial of multi-degree $(0, 4, 4, 4, 4, 1)$.
3. Replace every $b_i^1 b_j^2 b_k^3 b_l^4$ with $x_{i,j,k,l}$
   - Produce a polynomial of multi-degree $(0, 0, 4, 4, 4, 2)$.
4. Repeat for $c, d, e$,
   - Produce $P(x)$ of multi-degree $(0, 0, 0, 0, 0, 5)$ (possibly zero).
5. output: $P(x)$ highest weight vector of $S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3 \otimes S_{221}$.

# Evaluation of Young Symmetrizers

Multi-partition: (221, 221, 221,221), Filling: $F = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c & \\ \hline\end{array}$

Auxiliary polynomial:

$$p = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ b_1^1 & b_2^1 & b_3^1 \\ c_1^1 & c_2^1 & c_3^1 \end{vmatrix} \begin{vmatrix} d_1^1 & d_2^1 \\ e_1^1 & e_2^1 \end{vmatrix} \cdot \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix} \begin{vmatrix} d_1^2 & d_2^2 \\ e_1^2 & e_2^2 \end{vmatrix} \cdot \begin{vmatrix} a_1^3 & a_2^3 & a_3^3 \\ b_1^3 & b_2^3 & b_3^3 \\ c_1^3 & c_2^3 & c_3^3 \end{vmatrix} \begin{vmatrix} d_1^3 & d_2^3 \\ e_1^3 & e_2^3 \end{vmatrix} \cdot \begin{vmatrix} a_1^4 & a_2^4 & a_3^4 \\ b_1^4 & b_2^4 & b_3^4 \\ c_1^4 & c_2^4 & c_3^4 \end{vmatrix} \begin{vmatrix} d_1^4 & d_2^4 \\ e_1^4 & e_2^4 \end{vmatrix}$$

Point: $Z \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

1. Start with $p(a, b, c, d, e, x)$ of multi-degree $(4, 4, 4, 4, 4, 0)$.
2. Replace every $a_i^1 a_j^2 a_k^3 a_l^4$ with $x_{i,j,k,l}$ and substitute $x_{i,j,k,l} \to Z_{i,j,k,l}$.
   - Produce a polynomial of multi-degree $(0, 4, 4, 4, 4, 0)$.
3. Replace every $b_i^1 b_j^2 b_k^3 b_l^4$ with $x_{i,j,k,l}$ and substitute $x_{i,j,k,l} \to Z_{i,j,k,l}$.
   - Produce a polynomial of multi-degree $(0, 0, 4, 4, 4, 0)$.
4. Repeat for $c, d, e$,
5. output: the value of $p(Z)$.
   - Producing $p(Z)$ takes *much* less time and memory than $p(x)$.

# Compute a basis of $M_\pi$ for each $\pi$.

The following fillings form a basis of $M_{(221),(221),(221),(221)}$:

$$F_1 = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array}, \qquad F_2 = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d \\ \cline{1-1}\end{array}$$

$$F_3 = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline e \\ \cline{1-1}\end{array}, \qquad F_4 = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d \\ \cline{1-1}\end{array}$$

$$F_5 = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline e \\ \cline{1-1}\end{array}, \qquad F_6 = \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array}$$

Check $\operatorname{rank}(p_i(Z_j))$ for 6 random points $Z_j$, $\Rightarrow$ independence.

A character computation $\Rightarrow$ spanning.

# Evaluate the highest-weight space of $S_\pi V \otimes M_\pi$ on $Z$.

Using the basis of $M_{(221),(221),(221),(221)}$ and Young symmetrizers, populate the matrix (one processor core per entry)

$$(p_i(Z_j))$$

for 6 random points $Z_j$ of $Z$.

Find null-space (kernel) is the span of

$$\begin{pmatrix} -11/12 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}^T$$

So $(S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3 \otimes S_{221}\mathbb{C}^3)$ has multiplicity 1 in $I(Z)$.

# Quadrifocal Hilbert Function

Compute the Hilbert function for $I(Z) \subset R = \mathbb{C}[x_1, \ldots, x_{81}]$ for as high a degree as possible.

Using Representation Theory and parallel computing we found:

$$
\begin{array}{lcrrrrrrrrr}
d & = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
HF_R & = & (1 & 81 & 3321 & 91881 & 1929501 & 32801517 & 470155077 & 5843355957 & 64276915527 \\
\dim_I & = & (0 & 0 & 0 & 600 & 48600 & 1993977 & 54890407 & 1140730128 & 18051062139 \\
\text{mingens} & = & (0 & 0 & 0 & 600 & 0 & \geq 1377 & 37586 & ?0 & ?162000
\end{array}
$$

```
-- deg      # reps      max_mult
--------------------------------
-- 1 |           1           1           }
-- 2 |           3           1           } Use Symmetry
-- 3 |           9           3           }
-- 4 |          25           4           } Use Grid Computing
-- 5 |          59          13           }
-- 6 |         163          93           } Use Multi-threading
-- 7 |         288         301           }
-- 8 |         619         608           } Use a High Performance Cluster
```

# Obtain a list of $G$-modules (with multiplicity) in $I(Z)$.

Repeat the process for isotypic decompositions of $\mathbb{C}[(\mathbb{C}^3)^{\otimes 4}]_d$.

Obtain multiplicities of all modules in $I(Z)_d$ for small degree $d$.
For instance, $I(Z)_d = 0$ for $d = 1, 2$

Input the results into SchurRings (by Stillman and Raicu) in Macaualy2.

Modules are represented as polynomials, with coefficients the multiplicities.
For example $I(Z)_3$ is expressed as

$$\begin{gathered}(s_{(1,1,1)}t_{(1,1,1)}u_3 + (s_{(1,1,1)}t_3 + s_3 t_{(1,1,1)})u_{(1,1,1)})v_3 \\ +((s_{(1,1,1)}t_3 + s_3 t_{(1,1,1)})u_3 + s_3 t_3 u_{(1,1,1)})v_{(1,1,1)}\end{gathered}$$

or modding out by the $\mathfrak{S}_4$ action, $I(Z)_3 = s_3 t_3 u_{(1,1,1)} v_{(1,1,1)}$

which represents the module $\mathfrak{S}_4.(S_3\mathbb{C}^3 \otimes S_3\mathbb{C}^3 \otimes S_{1,1,1}\mathbb{C}^3 \otimes S_{1,1,1}\mathbb{C}^3)$.

# Determine which modules are minimal generators.

Let $R = \mathbb{C}[(\mathbb{C}^3)^{\otimes 4}]$. Using the Young symmetrizer algorithm,
$I(Z)_4 = (s_4 t_4 + (s_4 + s_{(3,1)})t_{(3,1)})u_{(2,1,1)}v_{(2,1,1)}$

Using `SchurRings`, we find that
$I(Z)_3 \cdot R_1 = I(Z)_4$.
(all multiplicities are one, and every module in $I(Z)_4$ occurs in $I(Z)_3 \cdot R_1$)
So there are no minimal generators in degree 4.

We find two modules in $I(Z)_5$ that cannot occur in $I(Z)_3 \cdot R_2$:
$s_{(3,1,1)}t_{(3,1,1)}u_{(3,1,1)}v_{(3,1,1)} + s_{(2,2,1)}t_{(2,2,1)}u_{(2,2,1)}v_{(2,2,1)}$

We find the following modules occur in $I(Z)_6$ but cannot occur in $I(Z)_5 \cdot R_1$:
$(s_6 t_{(3,3)}u_{(3,3)} + ((2s_{(4,1,1)} + 2s_{(3,3)})t_{(3,3)} + s_{(3,2,1)}t_{(3,2,1)} + 2s_{(2,2,2)}t_{(2,2,2)})u_{(2,2,2)})v_{(2,2,2)}$
We find that all modules in $I(Z)_7$ can occur in $I(Z)_6 \cdot R_1$,
strong evidence that there are no minimal generators in degree 7.
In degree 8 we find a surprise: $S_{4,4}S_{4,4}S_{4,4}S_{4,2,2} \otimes \mathbb{C}^2$ must occur among the minimal generators.
In degree 9 we weren't able to compute all modules because of a lack of computing time, but

| graded piece | dim $I_d$ | necessary $G$ modules of minimal generators | dimension of mingens |
|---|---|---|---|
| $I_2$ | 0 | $\mathcal{M}_2 = 0$ | 0 |
| $I_3$ | 600 | $\mathcal{M}_3 = S_3S_3S_{1,1,1}S_{1,1,1}$ | 600 |
| $I_4$ | 48,600 | $\mathcal{M}_4 = 0$ | 0 |
| $I_5$ | 1,993,977 | $\mathcal{M}_5 = S_{3,1,1}S_{3,1,1}S_{3,1,1}S_{3,1,1} \oplus S_{2,2,1}S_{2,2,1}S_{2,2,1}S_{2,2,1}$ | 1,377 |
| $I_6$ | 54,890,407 | $\mathcal{M}_6 = S_{4,1,1}S_{3,3}S_{2,2,2}S_{2,2,2}\otimes\mathbb{C}^2 \oplus S_{3,3}S_{3,3}S_{2,2,2}S_{2,2,2}\otimes\mathbb{C}^2 \oplus S_{3,2,1}S_{3,2,1}S_{2,2,2}S_{2,2,2} \oplus S_{2,2,2}S_{2,2,2}S_{2,2,2}S_{2,2,2}\otimes\mathbb{C}^2 \oplus S_6S_{3,3}S_{3,3}S_{2,2,2}$ | 37,586 |
| $I_7$ | 1,140,730,128 | $\mathcal{M}_7 = 0$ | 0 |
| $I_8$ | 18,051,062,139 | $\mathcal{M}_8 = S_{4,4}S_{4,4}S_{4,4}S_{4,2,2} \otimes \mathbb{C}^2$ | 162,000 |
| $I_9$ | $\geq$ 188,850,321,637 | $\mathcal{M}_9 \geq S_{5,4}S_{5,4}S_{5,4}S_{4,3,2} \oplus S_{5,4}S_{5,4}S_{5,4}S_{5,2,2}$ | 3,087,000 |

Table: The ideal of the quadrifocal variety up to degree 9. We used M2 to rule out many possible minimal generators and conjecture that these equations suffice to define the quadrifocal variety.