Math 2650 - Linear Differential Equations

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Separation of Variables

Separation of variables is an analytical method tha may be used to solve certain d.e.s (and certain i.v.p.s). At the heart of the method is the chain rule.

Consider a (first order) differential equation that may be written as

$$\frac{dx}{dt} = f(x)\, g(t).$$

That is, the derivative of the function is equal to the product of a function of <u>only</u> x and a function of <u>only</u> t (the r.h.s. of the equation).

If $f(x) \neq 0$ we can write

$$\frac{1}{f(x)} \frac{dx}{dt} = g(t).$$

Recall $x = x(t)$, so if we can find a function $H(x)$ with $\dfrac{dH}{dx} = \dfrac{1}{f(x)}$, using the chain rule we have that

$$\frac{d}{dt} H(x) = \frac{d}{dx} H(x) \frac{dx}{dt} = \frac{1}{f(x)} \frac{dx}{dt}$$

so

$$\frac{d}{dt} H(x) = g(t).$$

Integrating both sides of the equation w.r.t. t we get

$$H(x) = \int g(t)\, dt + c$$

and since

$$H(x) = \int \frac{1}{f(x)}\, dx,$$

the solution of the d.e. is given (implicitly) by

$$\int \frac{1}{f(x)}\, dx = \int g(t)\, dt + C.$$

Note: the above is an implicit solution. Also note the arbitrary constant C, this is the general solution. If we are also given an initial condition we can use it to solve for C and obtain a particular solution.

Examples:

1.
$$\frac{dx}{dt} = k\, x$$

if $x \neq 0$, seperating the variables

$$\frac{1}{x}\, dx = k\, dt$$

integrating

$$\ln(|\,x\,|) = k\, t + C$$

exponentiating

$$|\,x\,| = e^{(k\, t + C)}$$

so

$$|\,x\,| = C_1\, e^{(k\, t)} \qquad \text{where} \quad C_1 = e^{C}$$

and absorbing the sign into the constant C_1 we get the solution

$$x(t) = C_2\, e^{(k\, t)}$$

another solution is

$$x(t) = 0.$$

What is the interval of existence of these solutions?

Exercise: verify that the above are solutions of the differential equation.

2.

$$\frac{dx}{dt} = x^2 t$$

if $x \neq 0$

$$\frac{dx}{x^2} = t\, dt$$

a solution is given by

$$\int \frac{1}{x^2}\, dx = \int t\, dt + C$$

```
> restart;
> int(1/x^2,x);
```

$$-\frac{1}{x}$$

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> int(t,t);
```

$$\frac{1}{2} t^2$$

So a solution is

$$-\frac{1}{x} = \frac{t^2}{2} + C.$$

Solving for x

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> solve(-1/x = t^2/2+C,x);
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$$-\frac{2\,1}{t^2 + 2\,C}$$

we obtain

$$x(t) = -\frac{2}{t^2 + 2\,C}$$

or

$$x(t) = \frac{2}{C - t^2}.$$

another solution is
$$x(t) = 0.$$
What is the interval of existence of these solutions? What does it depend on?

Exercise: verify that the above are solutions of the differential equation.

3.
$$\frac{dx}{dt} = x^2 \sin(t) + x \sin(t)$$

rewrite the equation as
$$\frac{dx}{dt} = (x^2 + x) \sin(t)$$

if $x^2 + x \neq 0$
$$\frac{dx}{x^2 + x} = \sin(t) \, dt$$

a solution is given by
$$\int \frac{1}{x^2 + x} \, dx = \int \sin(t) \, dt + C$$

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> int(1/(x^2+x),x);
```
$$\ln(x) - \ln(x + 1)$$

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> int(sin(t),t);
```
$$-\cos(t)$$

So a solution is
$$\ln(x) - \ln(x + 1) = -\cos(t) + C$$

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> solve(ln(x)-ln(x+1) = -cos(t)+C,x);
```
$$\frac{1}{e^{(\cos(t) - C)} - 1}$$

so
$$x(t) = \frac{1}{C e^{\cos(t)} - 1}$$

```
> solve(x^2+x=0);
```
$$0, -1$$

so additional solutions are
$$x(t) = 0 \quad \text{and} \quad x(t) = -1.$$

What is the interval of existence of these solutions? What does it depend on?

Exercise: verify that the above are solutions of the differential equation.

Problem:

Solve the equation

$$\frac{dx}{dt} = \frac{x}{t}.$$