A GALERKIN METHOD FOR TIME-DEPENDENT MHD FLOW WITH NONIDEAL BOUNDARIES

Paul G. Schmidt

Department of Mathematics, Auburn University,
AL 36849-5310, USA

Abstract. A novel formulation is given for the equations of viscous incompressible magnetohydrodynamics with realistic ("nonideal") boundary conditions that account for the fluid's interaction with the outside world. The global-in-time existence of weak solutions is established via the Galerkin method.

AMS (MOS) Subject Classification. 76W05, 35Q30, 35Q60, 65M60

INTRODUCTION

Magnetohydrodynamics (or MHD) describes the flow of electrically conducting fluids in the presence of body forces, electric currents, and electromagnetic fields. Assuming the fluids to be viscous and incompressible, the governing equations are the Navier-Stokes and pre-Maxwell equations, coupled via the Lorentz force and Ohm's law (see, for example, [8, Chapter 2]).

While the Navier-Stokes equations are posed in the regions occupied by conducting fluids, Maxwell's equations are posed in all of space, and typically, both interior and exterior fields must be determined. Only under special circumstances, most notably in the presence of perfectly conducting walls ("ideal boundaries"), is it legitimate to confine attention to a single body of conducting fluid and to neglect its electromagnetic interaction with the outside world. In general, this interaction is of critical importance; it constitutes what mostly distinguishes MHD from ordinary hydrodynamics and is a source of challenging mathematical and computational problems. We refer to the introduction of [6] for further discussion and a review of the relevant mathematical literature.

Traditionally, the MHD equations are formulated as a system of evolution equations for the fluid velocity and the magnetic field, along with an auxiliary equation for the electric field outside the fluid region. See [3, 4] for seminal contributions based on this approach and [9] for a recent improvement. The fact that the magnetic field extends to all of space and may exhibit jump discontinuities across interfaces separating media with different electromagnetic properties necessitates the use of unusual and complicated function spaces. While the ensuing difficulties are analytically tractable, they all but preclude the application of standard numerical techniques, such as finite-element methods, for the numerical approximation of solutions.

In [5–7] we developed a novel approach to MHD that avoids some intrinsic difficulties of the traditional method by employing the electric current density rather than the magnetic field as the primary electromagnetic variable. Our "velocity-current formulation" exploits the simple fact that while electric and magnetic fields may extend throughout space, the unknown currents inducing those fields are typically carried by conductors of finite extent. If we consider, for example, a single body of conducting fluid and assume all external field sources to be known, the only unknown current flows in the fluid region itself. In general, the velocity-current formulation leads to a system of evolution equations for the fluid velocity and the unknown current density in the fluids and surrounding solid conductors, along with an auxiliary linear div-curl system in $\mathbf{R}^3$; the latter can frequently be solved analytically in terms of singular integrals.

The function spaces arising in this formulation are standard Lebesgue and Sobolev spaces, which allows the immediate application of finite-element based discretization techniques. In [5–7] this approach was successfully pursued in the mathematical analysis, numerical approximation, and computational simulation of various *stationary* MHD flow problems. Here we present, for the first time, an application to a full, *time-dependent* problem. To highlight the main ideas without overloading the exposition with technical details, we consider a rather simple scenario, with a single body of conducting fluid and known external field sources, and we only prove the global-in-time existence of weak solutions. The novelty is not so much in the *result*, which could also be obtained with the methods of [3, 4] or [9] (although our boundary conditions are more general), but in the *approach*. This approach is potentially applicable to a much wider class of problems; it yields results, analogous to those in [9], that go far beyond the existence of weak solutions; and most importantly, it allows the design of efficient algorithms for the numerical approximation of solutions. All these aspects are the subject of ongoing research and will be presented in more detail in forthcoming publications.

The present paper is organized as follows. In Section 1 we present the velocity-current formulation for the time-dependent MHD equations. In Section 2 we derive an abstract version of the problem in a Hilbert space setting. Finally, in Section 3, we establish the existence of global-in-time weak solutions via the Galerkin method.

1. THE PROBLEM AND ITS PHYSICAL BACKGROUND

We are concerned with the flow of a viscous, incompressible, electrically conducting fluid, confined to a bounded region of space and interacting with various body forces, electric currents, and electromagnetic fields. Roughly speaking, the flow is governed by the Navier-Stokes equations, posed in the fluid region Ω, coupled to Maxwell's equations, posed in all of $\mathbf{R}^3$. As discussed in the introduction, we seek to formulate the problem as a system of evolution equations for the fluid velocity $\mathbf{u}$ and the current density $\mathbf{J}$ in the fluid; both are solenoidal vector fields, depending on time $t \in (0,T)$ and position $x \in \Omega$.

For the velocity field we have

$$\rho\mathbf{u}_t - \eta\Delta\mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{J}\times\mathbf{B} = \mathbf{f} \quad \text{and} \quad \nabla\cdot\mathbf{u} = 0 \quad \text{in } (0,T)\times\Omega. \tag{1}$$

These are the familiar Navier-Stokes equations, except for the term $\mathbf{J}\times\mathbf{B}$, which represents the Lorentz force; $\mathbf{B}$ is the (unknown) magnetic field. The density ρ and viscosity η are given positive constants; $\mathbf{f}$ is a given body force; and the scalar pressure p is an auxiliary unknown that plays the role of a Lagrange multiplier associated with the constraint $\nabla\cdot\mathbf{u} = 0$.

The equations (1) must be supplemented by suitable boundary and initial conditions for $\mathbf{u}$, for example,

$$\mathbf{u} = \mathbf{g} \quad \text{on } (0,T)\times\partial\Omega \quad \text{and} \quad \mathbf{u} = \mathbf{u}_0 \quad \text{on } \{0\}\times\Omega. \tag{2}$$

Here $\mathbf{g}$ is a given vector field, tangential to the boundary of Ω, and $\mathbf{u}_0$ is a suitably chosen initial velocity field. By imposing a nonhomogeneous Dirichlet boundary condition, we allow the fluid to be mechanically driven through boundary forcing.

The current density $\mathbf{J}$ in the fluid obeys Ohm's law,

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}) \quad \text{in } (0,T)\times\Omega, \tag{3}$$

where the conductivity σ is a given positive constant and $\mathbf{E}$ denotes the (unknown) electric field. We allow additional currents $\mathbf{J}_{ext}$ to flow in external conductors, possibly connected to the fluid region via electrodes on the surface. These external currents are assumed to be given; they could be generated by an adjustable voltage source, somewhere in an external circuit, for the purpose of driving the fluid in Ω (this is the operating principle of an MHD propulsion device). The total current distribution,

$$\tilde{\mathbf{J}} = \begin{cases} \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}) & \text{in } (0,T)\times\Omega, \\ \mathbf{J}_{ext} & \text{in } (0,T)\times\mathbf{R}^3\setminus\overline{\Omega}, \end{cases} \tag{4}$$

must satisfy the continuity equation

$$\nabla\cdot\tilde{\mathbf{J}} = 0 \quad \text{in } (0,T)\times\mathbf{R}^3. \tag{5}$$

Assuming the given external current distribution $\mathbf{J}_{ext}$ to be solenoidal in $(0,T)\times\mathbf{R}^3\backslash\overline{\Omega}$, Equation (5) is equivalent to

$$\nabla\cdot\mathbf{J}=0 \quad \text{in } (0,T)\times\Omega \quad \text{and} \quad \mathbf{J}\cdot\mathbf{n}=\mathbf{J}_{ext}\cdot\mathbf{n} \quad \text{on } (0,T)\times\partial\Omega, \tag{6}$$

where $\mathbf{n}$ denotes the outward unit normal vector field on $\partial\Omega$.

We now proceed to "eliminate" the unknown vector fields $\mathbf{B}$ and $\mathbf{E}$. Our arguments will not be mathematically rigorous at this stage, but any gaps that remain will be closed in Section 2. To begin with, we write the magnetic field as

$$\mathbf{B}=\mathbf{B}_{ext}+\mathcal{B}(\tilde{\mathbf{J}})\,,$$

where $\mathbf{B}_{ext}$ is a given solenoidal vector field on $(0,T)\times\mathbf{R}^3$ (possibly induced by currents other than $\tilde{\mathbf{J}}$ or due to permanent magnetism), while $\mathcal{B}(\tilde{\mathbf{J}})$ is the field induced by $\tilde{\mathbf{J}}$. Adopting the quasi-stationary form of Maxwell's equations, as is the custom in MHD, we obtain $\mathcal{B}(\tilde{\mathbf{J}})$ as the solution of

$$\nabla\times\mu^{-1}\mathcal{B}(\tilde{\mathbf{J}})=\tilde{\mathbf{J}} \quad \text{and} \quad \nabla\cdot\mathcal{B}(\tilde{\mathbf{J}})=0\,,$$

where μ denotes the magnetic permeability. For simplicity we assume the fluid as well as all materials outside to be nonmagnetic, so that μ is the permeability of the vacuum (a positive constant).

Next we introduce solenoidal vector potentials $\mathbf{A}_{ext}$ and $\mathcal{A}(\tilde{\mathbf{J}})$ for the solenoidal vector fields $\mathbf{B}_{ext}$ and $\mathcal{B}(\tilde{\mathbf{J}})$, that is, we solve

$$\nabla\times\mathbf{A}_{ext}=\mathbf{B}_{ext} \quad \text{and} \quad \nabla\cdot\mathbf{A}_{ext}=0\,,$$

$$\nabla\times\mathcal{A}(\tilde{\mathbf{J}})=\mathcal{B}(\tilde{\mathbf{J}}) \quad \text{and} \quad \nabla\cdot\mathcal{A}(\tilde{\mathbf{J}})=0\,.$$

Since we have $\nabla\times\mu^{-1}\mathcal{B}(\tilde{\mathbf{J}})=\tilde{\mathbf{J}}$ and since μ is assumed to be constant, $\mathcal{A}(\tilde{\mathbf{J}})$ satisfies

$$\nabla\times\nabla\times\mathcal{A}(\tilde{\mathbf{J}})=\mu\tilde{\mathbf{J}} \quad \text{and} \quad \nabla\cdot\mathcal{A}(\tilde{\mathbf{J}})=0\,,$$

or equivalently, $-\Delta\mathcal{A}(\tilde{\mathbf{J}})=\mu\tilde{\mathbf{J}}$. That is, we have $\mathcal{A}(\tilde{\mathbf{J}})=\mu\mathcal{L}(\tilde{\mathbf{J}})$, where (formally) $\mathcal{L}=(-\Delta)^{-1}$. With similar reasoning we get $\mathbf{A}_{ext}=\mathcal{L}(\nabla\times\mathbf{B}_{ext})=\nabla\times\mathcal{L}(\mathbf{B}_{ext})$. (All this will be made mathematically precise in Section 2.)

The electric field $\mathbf{E}$ obeys Faraday's law,

$$\nabla\times\mathbf{E}=-\mathbf{B}_t \quad \text{in } (0,T)\times\mathbf{R}^3.$$

Since $\mathbf{B}=\nabla\times\mathbf{A}$ with $\mathbf{A}=\mathbf{A}_{ext}+\mathcal{A}(\tilde{\mathbf{J}})$, it follows that $\nabla\times(\mathbf{E}+\mathbf{A}_t)=0$. This allows us to introduce a scalar potential ϕ such that $\mathbf{E}+\mathbf{A}_t=-\nabla\phi$, that is, $\mathbf{E}=-\mathbf{A}_t-\nabla\phi$. But $\mathbf{A}_t=\mathbf{A}_{ext,t}+\mathcal{A}(\tilde{\mathbf{J}})_t=\mathbf{A}_{ext,t}+\mu\mathcal{L}(\tilde{\mathbf{J}}_t)$ and thus,

$$\mathbf{E}=\mathbf{E}_{ext}-\mu\mathcal{L}(\tilde{\mathbf{J}}_t)-\nabla\phi\,,$$

where $\mathbf{E}_{ext} = -\mathbf{A}_{ext,t} = -\nabla \times \mathcal{L}(\mathbf{B}_{ext,t})$. Substituting this into Ohm's law (3), we obtain

$$\mathbf{J} = \sigma(\mathbf{E}_{ext} - \mu\mathcal{L}(\tilde{\mathbf{J}}_t) - \nabla\phi + \mathbf{u} \times \mathbf{B}) \quad \text{in } (0,T) \times \Omega.$$

Together with (5), this leads to the following equations for the current density $\mathbf{J}$ in the fluid region:

$$\mu\mathcal{L}(\tilde{\mathbf{J}}_t) + \sigma^{-1}\mathbf{J} + \nabla\phi - \mathbf{u} \times \mathbf{B} = \mathbf{E}_{ext} \quad \text{and} \quad \nabla \cdot \mathbf{J} = 0 \quad \text{in } (0,T) \times \Omega. \tag{7}$$

Note that the scalar potential ϕ plays a role analogous to that of the pressure p in the Navier-Stokes equations (1). We supplement (7) with the boundary condition from (6) and a suitable initial condition for $\mathbf{J}$:

$$\mathbf{J} \cdot \mathbf{n} = \mathbf{J}_{ext} \cdot \mathbf{n} \quad \text{on } (0,T) \times \partial\Omega \quad \text{and} \quad \mathbf{J} = \mathbf{J}_0 \quad \text{on } \{0\} \times \Omega. \tag{8}$$

Summarizing, our problem is the following: Given the region Ω, parameters ρ, η, σ, and μ, data $\mathbf{f}$, $\mathbf{g}$, $\mathbf{J}_{ext}$, $\mathbf{B}_{ext}$, and $\mathbf{E}_{ext}$, and initial values $\mathbf{u}_0$ and $\mathbf{J}_0$, find vector fields $\mathbf{u}$ and $\mathbf{J}$ and scalar fields p and ϕ on $(0,T) \times \Omega$ such that Equations (1), (2), (7), and (8) are satisfied (at least in a weak sense), with $\mathbf{B}$ given by

$$\mathbf{B} = \mathbf{B}_{ext} + \mathcal{B}(\tilde{\mathbf{J}}) = \mathbf{B}_{ext} + \mu\nabla \times \mathcal{L}(\tilde{\mathbf{J}})$$

and $\tilde{\mathbf{J}} = \mathbf{J} + \mathbf{J}_{ext}$ (that is, $\tilde{\mathbf{J}}|_{(0,T)\times\Omega} = \mathbf{J}$ and $\tilde{\mathbf{J}}|_{(0,T)\times\mathbf{R}^3\setminus\overline{\Omega}} = \mathbf{J}_{ext}$).

All our subsequent considerations will be based on the following list of assumptions.

Standing Assumptions.

(a) The fluid region Ω is a bounded domain in $\mathbf{R}^3$, with Lipschitz continuous boundary $\partial\Omega$ and outward unit normal vector field $\mathbf{n}$, and T is a fixed positive number.

(b) The parameters ρ, η, σ, and μ are given positive constants.

(c) The boundary velocity $\mathbf{g}$ is the restriction of a C^1-vector field $\mathbf{u}^*$ on $[0,T] \times \overline{\Omega}$ with $\nabla \cdot \mathbf{u}^* = 0$ in $(0,T) \times \Omega$ and $\mathbf{u}^* \cdot \mathbf{n} = 0$ on $(0,T) \times \partial\Omega$.

(d) The external current distribution $\mathbf{J}_{ext}$ is the restriction of a C^1-vector field $\mathbf{J}^*$ on $[0,T] \times \mathbf{R}^3$ with compact support and $\nabla \cdot \mathbf{J}^* = 0$ in $(0,T) \times \mathbf{R}^3$.

(e) The initial velocity $\mathbf{u}_0$ is a C^1-vector field on $\overline{\Omega}$, satisfying $\nabla \cdot \mathbf{u}_0 = 0$ in Ω and $\mathbf{u}_0 = \mathbf{g}(0,\cdot)$ on $\partial\Omega$.

(f) The initial current density $\mathbf{J}_0$ is a C^1-vector field on $\overline{\Omega}$, satisfying $\nabla \cdot \mathbf{J}_0 = 0$ in Ω and $\mathbf{J}_0 \cdot \mathbf{n} = \mathbf{J}_{ext}(0,\cdot) \cdot \mathbf{n}$ on $\partial\Omega$.

(g) The body force $\mathbf{f}$ is a continuous vector field on $[0,T] \times \overline{\Omega}$.

(h) The external magnetic and electric fields $\mathbf{B}_{ext}$ and $\mathbf{E}_{ext}$ are continuous vector fields on $[0,T] \times \mathbf{R}^3$.

Regarding the last assumption, recall that in terms of the underlying physics, $\mathbf{B}_{ext}$ and $\mathbf{E}_{ext}$ are *not* independent data — in fact, $\mathbf{E}_{ext}$ is the electric field induced by the time variation of $\mathbf{B}_{ext}$. Also, both fields should be solenoidal. For our present purposes, though, none of the above is relevant.

We note that Assumptions (c)–(h) could be relaxed considerably. In particular, since our goal is to derive a *weak* formulation of the problem and to establish the existence of *weak* solutions, none of the data would need to be differentiable in the classical sense. However, the above set of assumptions is convenient without being overly restrictive, and maximum generality is anyway not our objective here.

2. ABSTRACT FORMULATION OF THE PROBLEM

Throughout, we will use standard notation for L^p-spaces (with $1 \leq p \leq \infty$) and for spaces of continuous or continuously differentiable functions. For any open subset D of $\mathbf{R}^3$ and any positive integer m, we denote by $H^m(D)$ the usual Sobolev space of square-integrable functions on D with square-integrable derivatives up to order m. By $H_0^m(D)$ we mean the closure of $C_0^\infty(D)$ in $H^m(D)$ and by $H^{-m}(D)$, the norm dual of $H_0^m(D)$.

We will also need the Beppo-Levi space $W^m(\mathbf{R}^3)$, defined as the completion of $C_0^\infty(\mathbf{R}^3)$ with respect to the norm $f \mapsto \left(\sum_{|\alpha|=m} \|D^\alpha f\|^2_{L^2(\mathbf{R}^3)}\right)^{1/2}$, and its norm dual $W^{-m}(\mathbf{R}^3)$; the latter is a space of distributions on $\mathbf{R}^3$. It is well known (see, for example, [1, Vol. 4, Chapter XI.B.1]) that $H^m(\mathbf{R}^3) \hookrightarrow W^m(\mathbf{R}^3) \hookrightarrow H^m_{loc}(\mathbf{R}^3)$, with continuous embeddings.

Spaces of vector (that is, $\mathbf{R}^3$-valued) functions or distributions will be distinguished by bold-face type, so that, for example, $\mathbf{L}^2(\mathbf{R}^3) = (L^2(\mathbf{R}^3))^3$. For any space $\mathbf{X}(D)$ of vector functions or distributions on D, the symbol $\mathbf{X}_{div}(D)$ will denote the subspace of all divergence-free (or solenoidal) members of $\mathbf{X}(D)$.

We write the inner product in any Hilbert space X as $\langle \cdot, \cdot \rangle_X$ and the duality pairing between any Banach space Y and its norm dual Y^* as $\langle \cdot, \cdot \rangle_{Y^*,Y}$. Given Banach spaces X, Y, and Z, the symbol $[X, Y]$ stands for the space of all bounded linear operators from X into Y, the symbol $[X, Y; Z]$ for the space of all bounded bilinear operators from $X \times Y$ into Z.

Consider the Hilbert space $\mathbf{W}^1(\mathbf{R}^3)$, whose natural inner product is given by

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathbf{W}^1(\mathbf{R}^3)} = \int_{\mathbf{R}^3} (\nabla \mathbf{f}) \cdot (\nabla \mathbf{g}) = \int_{\mathbf{R}^3} \left((\nabla \times \mathbf{f}) \cdot (\nabla \times \mathbf{g}) + (\nabla \cdot \mathbf{f})(\nabla \cdot \mathbf{g})\right). \quad (9)$$

(The first equality is a definition; the second one is easily verified for $\mathbf{f}, \mathbf{g} \in \mathbf{C}_0^\infty(\mathbf{R}^3)$, from which the general case follows by a density argument.) Define a bounded linear isomorphism $\mathcal{L}_1 : \mathbf{W}^{-1}(\mathbf{R}^3) \to \mathbf{W}^1(\mathbf{R}^3)$ by

$$\langle \mathcal{L}_1 \mathbf{f}, \mathbf{g} \rangle_{\mathbf{W}^1(\mathbf{R}^3)} = \langle \mathbf{f}, \mathbf{g} \rangle_{\mathbf{W}^{-1}(\mathbf{R}^3), \mathbf{W}^1(\mathbf{R}^3)}\,, \tag{10}$$

for $\mathbf{f} \in \mathbf{W}^{-1}(\mathbf{R}^3)$ and $\mathbf{g} \in \mathbf{W}^1(\mathbf{R}^3)$. For every $\mathbf{f} \in \mathbf{W}^{-1}(\mathbf{R}^3)$, the function $\mathbf{u} = \mathcal{L}_1 \mathbf{f}$ is the unique (weak) solution, in $\mathbf{W}^1(\mathbf{R}^3)$, of Poisson's equation, $-\Delta \mathbf{u} = \mathbf{f}$. Moreover, if $\mathbf{f} \in \mathbf{W}^{-1}_{div}(\mathbf{R}^3)$, then $\mathbf{u} = \mathcal{L}_1 \mathbf{f} \in \mathbf{W}^1_{div}(\mathbf{R}^3)$ and $-\Delta \mathbf{u} = \nabla \times \nabla \times \mathbf{u} = \mathbf{f}$ (in the weak sense). In fact, $\mathcal{L}_1$ restricts to an isomorphism between $\mathbf{W}^{-1}_{div}(\mathbf{R}^3)$ and $\mathbf{W}^1_{div}(\mathbf{R}^3)$.

It is well known (see, for example, [1, Vol. 1, Chapter II.3.1]) that if $\mathbf{f}$ is a distribution on $\mathbf{R}^3$ *with compact support*, then the equation $-\Delta \mathbf{u} = \mathbf{f}$ has a unique distributional solution that vanishes at infinity, namely, the Newtonian potential, $\mathbf{u} = G * \mathbf{f}$, where $G(x) = (4\pi|x|)^{-1}$ for $x \in \mathbf{R}^3 \setminus \{0\}$. If $\mathbf{f}$ belongs to $\mathbf{W}^{-1}(\mathbf{R}^3)$ and has compact support, then $G * \mathbf{f} \in \mathbf{W}^1(\mathbf{R}^3)$ and thus, $G * \mathbf{f} = \mathcal{L}_1 \mathbf{f}$. That is, the operator $\mathcal{L}_1$ is an extension of the Newtonian potential. Furthermore, if $\mathbf{f}$ belongs to $\mathbf{L}^2(\mathbf{R}^3)$ and has compact support, then $G * \mathbf{f} \in \mathbf{W}^2(\mathbf{R}^3)$, and as a consequence of the Calderon-Zygmund theorem, the mapping $\mathbf{f} \mapsto G * \mathbf{f}$ extends to a bounded linear isomorphism $\mathcal{L}_2 : \mathbf{L}^2(\mathbf{R}^3) \to \mathbf{W}^2(\mathbf{R}^3)$ (see, for example, [1, Vol. 1, Chapter II.3.2]). For every $\mathbf{f} \in \mathbf{L}^2(\mathbf{R}^3)$, the function $\mathbf{u} = \mathcal{L}_2 \mathbf{f}$ is the unique solution, in $\mathbf{W}^2(\mathbf{R}^3)$, of $-\Delta \mathbf{u} = \mathbf{f}$. Moreover, if $\mathbf{f} \in \mathbf{L}^2_{div}(\mathbf{R}^3)$, then $\mathbf{u} = \mathcal{L}_2 \mathbf{f} \in \mathbf{W}^2_{div}(\mathbf{R}^3)$ and $-\Delta \mathbf{u} = \nabla \times \nabla \times \mathbf{u} = \mathbf{f}$. In fact, $\mathcal{L}_2$ restricts to an isomorphism between $\mathbf{L}^2_{div}(\mathbf{R}^3)$ and $\mathbf{W}^2_{div}(\mathbf{R}^3)$.

Obviously, both operators, $\mathcal{L}_1$ and $\mathcal{L}_2$, are just different realizations of $(-\Delta)^{-1}$. In particular, both are extensions of the Newtonian potential and thus coincide on the intersection of their respective domains of definition. (Note that $\mathbf{L}^2(\mathbf{R}^3)$ does not embed into $\mathbf{W}^{-1}(\mathbf{R}^3)$, but that $\mathbf{W}^{-1}(\mathbf{R}^3)$ contains all functions $\mathbf{f} \in \mathbf{L}^2(\mathbf{R}^3)$ *with compact support*.) Abusing notation, we will henceforth denote both operators, $\mathcal{L}_1$ and $\mathcal{L}_2$, by the same symbol, $\mathcal{L}$. Also, we will frequently apply $\mathcal{L}$ to functions $\mathbf{f}$ that are defined and square-integrable on some bounded subdomain of $\mathbf{R}^3$, with the understanding that $\mathcal{L}\mathbf{f}$ then really means $\mathcal{L}\tilde{\mathbf{f}}$, where $\tilde{\mathbf{f}}$ stands for the zero extension of $\mathbf{f}$ to $\mathbf{R}^3$. We note that in this case, $\mathcal{L}\mathbf{f} \in \mathbf{W}^1(\mathbf{R}^3) \cap \mathbf{W}^2(\mathbf{R}^3)$, so that the derivatives of $\mathcal{L}\mathbf{f}$ belong to $\mathbf{H}^1(\mathbf{R}^3)$.

Now we are in a position to rigorously define the vector potential and magnetic field operators $\mathcal{A}$ and $\mathcal{B}$ needed for the velocity-current formulation of the MHD equations: Given the magnetic permeability μ, let

$$\mathcal{A} = \mu \mathcal{L} \quad \text{and} \quad \mathcal{B} = \nabla \times \mathcal{A} = \mu \nabla \times \mathcal{L}\,.$$

The following lemma gathers the properties of $\mathcal{A}$ and $\mathcal{B}$ that will be needed in the sequel. Everything follows readily from the corresponding properties of $\mathcal{L}$, as discussed above. For Part (c) of the lemma, recall (9) and (10).

Lemma 1.

(a) *The operator* $\mathcal{A}$ *is a bounded linear isomorphism between the spaces* $\mathbf{W}^{-1}_{(div)}(\mathbf{R}^3)$ *and* $\mathbf{W}^{1}_{(div)}(\mathbf{R}^3)$ *as well as between* $\mathbf{L}^{2}_{(div)}(\mathbf{R}^3)$ *and* $\mathbf{W}^{2}_{(div)}(\mathbf{R}^3)$*. The operator* $\mathcal{B}$ *is a bounded linear isomorphism between* $\mathbf{W}^{-1}_{div}(\mathbf{R}^3)$ *and* $\mathbf{L}^{2}_{div}(\mathbf{R}^3)$ *as well as between* $\mathbf{L}^{2}_{div}(\mathbf{R}^3)$ *and* $\mathbf{W}^{1}_{div}(\mathbf{R}^3)$*.*

(b) *For every* $\mathbf{J} \in \mathbf{W}^{-1}_{div}(\mathbf{R}^3)$ *the vector field* $\mathbf{A} = \mathcal{A}(\mathbf{J})$ *is the unique solution, in* $\mathbf{W}^1(\mathbf{R}^3)$*, of* $\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J}$ *and* $\nabla \cdot \mathbf{A} = 0$*. The vector field* $\mathbf{B} = \mathcal{B}(\mathbf{J})$ *is the unique solution, in* $\mathbf{L}^2(\mathbf{R}^3)$*, of* $\nabla \times \mathbf{B} = \mu \mathbf{J}$ *and* $\nabla \cdot \mathbf{B} = 0$*.*

(c) *For all* $\mathbf{J}, \mathbf{K} \in \mathbf{W}^{-1}(\mathbf{R}^3)$ *with* $\nabla \cdot \mathbf{J} = 0$ *or* $\nabla \cdot \mathbf{K} = 0$*, we have*

$$\langle \mathbf{J}, \mathcal{A}(\mathbf{K}) \rangle_{\mathbf{W}^{-1}(\mathbf{R}^3), \mathbf{W}^1(\mathbf{R}^3)} = \mu^{-1} \langle \mathcal{B}(\mathbf{J}), \mathcal{B}(\mathbf{K}) \rangle_{\mathbf{L}^2(\mathbf{R}^3)} .$$

(d) *If* $\mathbf{J} \in \mathbf{L}^2(\Omega)$ *with* $\nabla \cdot \mathbf{J} = 0$ *in* Ω *and* $\mathbf{J} \cdot \mathbf{n} = 0$ *on* $\partial\Omega$ *(where* Ω *is the bounded Lipschitz domain of Assumption (a) in Section 1), then the zero extension of* $\mathbf{J}$ *belongs to* $\mathbf{W}^{-1}_{div}(\mathbf{R}^3)$*, and for every* $\mathbf{K} \in \mathbf{W}^{-1}(\mathbf{R}^3)$ *we have*

$$\int_\Omega \mathbf{J} \cdot \mathcal{A}(\mathbf{K}) = \mu^{-1} \int_{\mathbf{R}^3} \mathcal{B}(\mathbf{J}) \cdot \mathcal{B}(\mathbf{K}) .$$

Lemma 1 provides rigorous justification for the velocity-current formulation of the MHD equations derived in Section 1. We are now ready to give a precise statement of the problem to be solved.

Problem (P). Under the assumptions (a)–(h) of Section 1, find $\mathbf{u} \in L^2(0,T;\mathbf{H}^1(\Omega))$, $\mathbf{J} \in L^2(0,T;\mathbf{L}^2(\Omega))$, $p \in L^1(0,T;L^2(\Omega))$, and $\phi \in L^1(0,T;H^1(\Omega))$ such that, in the sense of distributions,

$$\rho \mathbf{u}_t - \eta \Delta \mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \mathbf{J} \times \mathbf{B} = \mathbf{f} \quad \text{in } (0,T) \times \Omega, \tag{11}$$

$$\mu \mathcal{L}(\tilde{\mathbf{J}}_t) + \sigma^{-1} \mathbf{J} + \nabla \phi - \mathbf{u} \times \mathbf{B} = \mathbf{E}_{ext} \quad \text{in } (0,T) \times \Omega, \tag{12}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{J} = 0 \quad \text{in } (0,T) \times \Omega, \tag{13}$$

$$\mathbf{u} = \mathbf{g} \quad \text{and} \quad \mathbf{J} \cdot \mathbf{n} = \mathbf{J}_{ext} \cdot \mathbf{n} \quad \text{on } (0,T) \times \partial\Omega, \tag{14}$$

$$\mathbf{u} = \mathbf{u}_0 \quad \text{and} \quad \mathbf{J} = \mathbf{J}_0 \quad \text{on } \{0\} \times \Omega, \tag{15}$$

where $\mathbf{B} = \mathbf{B}_{ext} + \mu \nabla \times \mathcal{L}(\tilde{\mathbf{J}})$ and $\tilde{\mathbf{J}} = \mathbf{J} + \mathbf{J}_{ext}$.

Next, we introduce a number of test function spaces. Let $\mathcal{V}_1$ denote the set of all solenoidal vector fields in $\mathbf{C}_0^\infty(\Omega)$ and let V_1 and H_1 denote the closures of $\mathcal{V}_1$ in $\mathbf{H}^1(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively. That is,

$$V_1 = \{ \mathbf{v} \in \mathbf{H}^1(\Omega) \mid \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} = 0 \text{ on } \partial\Omega \} ,$$

$$H_1 = \{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \} .$$

(These are the test function spaces associated with the fluid velocity $\mathbf{u}$.) Let $\mathcal{V}_2 = \mathcal{V}_1$ and let V_2 and H_2 denote the closures of $\mathcal{V}_2$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{W}^{-1}(\mathbf{R}^3)$, respectively. That is,

$$V_2 = \{\mathbf{K} \in \mathbf{L}^2(\Omega) \mid \nabla \cdot \mathbf{K} = 0 \text{ in } \Omega,\ \mathbf{K} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,,$$

$$H_2 = \{\mathbf{K} \in \mathbf{W}^{-1}(\mathbf{R}^3) \mid \nabla \cdot \mathbf{K} = 0 \text{ in } \mathbf{R}^3,\ \mathbf{K} = 0 \text{ on } \mathbf{R}^3 \setminus \overline{\Omega}\}\,.$$

(These are the test function spaces associated with the current density $\mathbf{J}$.) Finally, let

$$\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2, \quad V = V_1 \times V_2, \quad \text{and} \quad H = H_1 \times H_2\,.$$

As subspaces of $\mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(\Omega) \times \mathbf{W}^{-1}(\mathbf{R}^3)$, respectively, both V and H are Hilbert spaces, and V is compactly and densely embedded in H.

To obtain a weak formulation of Problem (P), multiply Equations (11) and (12) by $\mathbf{v} \in V_1$ and $\mathbf{K} \in V_2$, respectively, then integrate over Ω and sum up. After some regrouping of terms, several integrations by part, and an application of Lemma 1(d), we arrive at a variational equation of the form

$$\begin{aligned} \frac{d}{dt} e((\mathbf{u}, \tilde{\mathbf{J}}), (\mathbf{v}, \mathbf{K})) &+ a_0((\mathbf{u}, \mathbf{J}), (\mathbf{v}, \mathbf{K})) \\ &+ b_0((\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{J}), (\mathbf{v}, \mathbf{K})) = \int_\Omega \mathbf{f} \cdot \mathbf{v} + \int_\Omega \mathbf{E}_{ext} \cdot \mathbf{K}\,. \end{aligned} \tag{16}$$

Here e and a_0 are bilinear forms, defined by

$$e((\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2)) = \rho \int_\Omega \mathbf{v}_1 \cdot \mathbf{v}_2 + \mu^{-1} \int_{\mathbf{R}^3} \mathcal{B}(\mathbf{K}_1) \cdot \mathcal{B}(\mathbf{K}_2)\,, \tag{17}$$

for $(\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2) \in \mathbf{L}^2(\Omega) \times \mathbf{W}^{-1}(\mathbf{R}^3)$, and

$$a_0((\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2)) = \eta \int_\Omega (\nabla \mathbf{v}_1) \cdot (\nabla \mathbf{v}_2) + \sigma^{-1} \int_\Omega \mathbf{K}_1 \cdot \mathbf{K}_2\,, \tag{18}$$

for $(\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2) \in \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$, while b_0 is a trilinear form, defined by

$$\begin{aligned} b_0((\mathbf{v}_0, \mathbf{B}_0), (\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2)) &= \rho \int_\Omega ((\mathbf{v}_0 \cdot \nabla)\mathbf{v}_1) \cdot \mathbf{v}_2 \\ &\quad + \int_\Omega ((\mathbf{K}_2 \times \mathbf{B}_0) \cdot \mathbf{v}_1 - (\mathbf{K}_1 \times \mathbf{B}_0) \cdot \mathbf{v}_2)\,, \end{aligned} \tag{19}$$

for $(\mathbf{v}_0, \mathbf{B}_0) \in \mathbf{L}^3(\Omega) \times \mathbf{L}^3(\Omega)$ and $(\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2) \in \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$. All three forms are bounded with respect to the product norms on their respective domains. (As for b_0, observe that $\mathbf{H}^1(\Omega)$ embeds continuously into $\mathbf{L}^6(\Omega)$ and use Hölder's inequality.)

For future reference we point out that

$$b_0((\mathbf{v}_0, \mathbf{B}_0), (\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2)) = -b_0((\mathbf{v}_0, \mathbf{B}_0), (\mathbf{v}_2, \mathbf{K}_2), (\mathbf{v}_1, \mathbf{K}_1))\,, \tag{20}$$

provided that $\mathbf{v}_0$ is solenoidal in Ω and that $\mathbf{v}_1$ or $\mathbf{v}_2$ vanishes on $\partial\Omega$. Finally we note that b_0, as given in (19), is also well defined (and bounded with respect to the appropriate norms) for $(\mathbf{v}_0, \mathbf{B}_0) \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$, $(\mathbf{v}_1, \mathbf{K}_1) \in \mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$, and $(\mathbf{v}_2, \mathbf{K}_2) \in \mathbf{L}^\infty(\Omega) \times \mathbf{L}^3(\Omega)$.

Using the liftings $\mathbf{u}^*$ and $\mathbf{J}^*$ of the boundary data $\mathbf{g}$ and $\mathbf{J}_{ext}$ (recall the assumptions (c) and (d) of Section 1), we now substitute $\mathbf{u} = \mathbf{u}^* + \hat{\mathbf{u}}$ and $\tilde{\mathbf{J}} = \mathbf{J}^* + \hat{\mathbf{J}}$ in Equation (16) and obtain the following variational equation for the new unknowns $\hat{\mathbf{u}} \in L^2(0,T;V_1)$ and $\hat{\mathbf{J}} \in L^2(0,T;V_2)$:

$$\begin{aligned}\frac{d}{dt}e((\hat{\mathbf{u}}, \hat{\mathbf{J}}), (\mathbf{v}, \mathbf{K})) + a_0((\hat{\mathbf{u}}, \hat{\mathbf{J}}), (\mathbf{v}, \mathbf{K})) + a(t, (\hat{\mathbf{u}}, \hat{\mathbf{J}}), (\mathbf{v}, \mathbf{K}))& \\ + b((\hat{\mathbf{u}}, \hat{\mathbf{J}}), (\hat{\mathbf{u}}, \hat{\mathbf{J}}), (\mathbf{v}, \mathbf{K})) = \ell(t, (\mathbf{v}, \mathbf{K}))\,.&\end{aligned} \tag{21}$$

Here b is a bounded trilinear form on $V \times V \times V$, defined by

$$b((\mathbf{v}_0, \mathbf{K}_0), (\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2)) = b_0((\mathbf{v}_0, \mathcal{B}(\mathbf{K}_0)), (\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2))\,, \tag{22}$$

while $a(t, \cdot, \cdot)$ and $\ell(t, \cdot)$, for $t \in (0,T)$, are bounded bilinear and linear forms on $V \times V$ and V, respectively, defined by

$$\begin{aligned}a(t, (\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2)) = b_0((\mathbf{u}^*(t), \mathbf{B}_{ext}(t) + \mathcal{B}(\mathbf{J}^*(t))), (\mathbf{v}_1, \mathbf{K}_1), (\mathbf{v}_2, \mathbf{K}_2))& \\ + b_0((\mathbf{v}_1, \mathcal{B}(\mathbf{K}_1)), (\mathbf{u}^*(t), \mathbf{J}^*(t)), (\mathbf{v}_2, \mathbf{K}_2))&\end{aligned} \tag{23}$$

and

$$\begin{aligned}\ell(t, (\mathbf{v}, \mathbf{K})) = \int_\Omega \mathbf{f}(t) \cdot \mathbf{v} + \int_\Omega \mathbf{E}_{ext}(t) \cdot \mathbf{K}& \\ - e((\mathbf{u}_t^*(t), \mathbf{J}_t^*(t)), (\mathbf{v}, \mathbf{K})) - a_0((\mathbf{u}^*(t), \mathbf{J}^*(t)), (\mathbf{v}, \mathbf{K}))& \\ - b_0((\mathbf{u}^*(t), \mathbf{B}_{ext}(t) + \mathcal{B}(\mathbf{J}^*(t))), (\mathbf{u}^*(t), \mathbf{J}^*(t)), (\mathbf{v}, \mathbf{K}))\,.&\end{aligned} \tag{24}$$

Regarding the boundedness of the trilinear form b on $V \times V \times V$, recall that the operator $\mathcal{B}$ is bounded from $\mathbf{L}^2(\mathbf{R}^3)$ into $\mathbf{W}^1(\mathbf{R}^3)$ and hence, from $\mathbf{L}^2(\Omega)$ into $\mathbf{H}^1(\Omega)$ (when composed with zero extension and restriction operators). Since $\mathbf{H}^1(\Omega)$ embeds continuously into $\mathbf{L}^3(\Omega)$, this shows in fact that b, as given in (22), is well defined and bounded not only on $V \times V \times V$, but on the triple product of $\mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$. Moreover, since $\mathcal{B}$ maps $\mathbf{W}^{-1}(\mathbf{R}^3)$ continuously into $\mathbf{L}^2(\mathbf{R}^3)$, b is also well defined and bounded on the product of $\mathbf{L}^2(\Omega) \times \mathbf{W}^{-1}(\mathbf{R}^3)$, $\mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$, and $\mathbf{L}^\infty(\Omega) \times \mathbf{L}^3(\Omega)$.

To simplify notation, we now let $u = (\hat{\mathbf{u}}, \hat{\mathbf{J}})$ and $v = (\mathbf{v}, \mathbf{K})$ and write Equation (21) as

$$\frac{d}{dt}e(u,v) + a_0(u,v) + a(t,u,v) + b(u,u,v) = \ell(t,v)\,. \tag{25}$$

Next we note that the bilinear forms e and a_0, as defined in (17) and (18), restrict to *inner products* on H and V, respectively, equivalent to the natural inner products that these spaces inherit from $\mathbf{L}^2(\Omega) \times \mathbf{W}^{-1}(\mathbf{R}^3)$ and $\mathbf{H}^1(\Omega) \times \mathbf{L}^2(\Omega)$. (As for H_2, recall from Lemma 1 that $\mathcal{B}$ is a bounded linear isomorphism between the spaces $\mathbf{W}^{-1}_{div}(\mathbf{R}^3)$ and $\mathbf{L}^2_{div}(\mathbf{R}^3)$; in fact, $\mu^{-1}\langle \mathcal{B}(\mathbf{K}_1), \mathcal{B}(\mathbf{K}_2)\rangle_{\mathbf{L}^2(\mathbf{R}^3)} = \mu\langle \mathbf{K}_1, \mathbf{K}_2\rangle_{\mathbf{W}^{-1}(\mathbf{R}^3)}$ for all $\mathbf{K}_1, \mathbf{K}_2 \in \mathbf{W}^{-1}_{div}(\mathbf{R}^3)$. As for V_1, apply Poincare's inequality.)

We will henceforth consider H and V as Hilbert spaces endowed with the inner products

$$\langle \cdot, \cdot \rangle_H = e|_{H \times H} \quad \text{and} \quad \langle \cdot, \cdot \rangle_V = a_0|_{V \times V} \, .$$

Moreover, we will identify H and its norm dual H^*, so that we have

$$V \hookrightarrow H = H^* \hookrightarrow V^*,$$

with compact dense embeddings. With this identification, we can write

$$e(v_1, v_2) = \langle v_1, v_2 \rangle_H = \langle v_1, v_2 \rangle_{V^*,V} \, ,$$

for $v_1 \in H$ and $v_2 \in V$. In addition, we introduce a linear isomorphism $L \in [V, V^*]$, a mapping $A \in L^2(0, T; [V, V^*])$, and a bilinear operator $B \in [V, V; V^*]$ such that

$$a_0(v_1, v_2) = \langle v_1, v_2 \rangle_V = \langle Lv_1, v_2 \rangle_{V^*,V} \, ,$$

$$a(t, v_1, v_2) = \langle A(t)v_1, v_2 \rangle_{V^*,V} \, ,$$

$$b(v_0, v_1, v_2) = \langle B(v_0, v_1), v_2 \rangle_{V^*,V} \, ,$$

for all $v_0, v_1, v_2 \in V$ and $t \in (0, T)$. Finally, we write

$$\ell(t, v) = \langle \ell(t), v \rangle_{V^*,V} \, ,$$

for $v \in V$ and $t \in (0, T)$. This allows us to replace Equation (25), which is to be satisfied for all $v \in V$, by the abstract evolution equation

$$\dot{u} + Lu + A(t)u + B(u, u) = \ell(t) \, .$$

Moreover, the initial conditions (15) for the original unknowns $\mathbf{u} \in L^2(0, T; \mathbf{H}^1(\Omega))$ and $\mathbf{J} \in L^2(0, T; \mathbf{L}^2(\Omega))$ are equivalent to an initial condition of the form $u(0) = u_0$ for the new unknown $u = (\hat{\mathbf{u}}, \hat{\mathbf{J}}) = (\mathbf{u} - \mathbf{u}^*, \mathbf{J} - \mathbf{J}^*) \in L^2(0, T; V)$. The assumptions (c)–(h) of Section 1 certainly guarantee that $u_0 = (\mathbf{u}_0 - \mathbf{u}^*(0), \mathbf{J}_0 - \mathbf{J}^*(0)) \in H$ and that $\ell \in L^2(0, T; V^*)$. We thus arrive at the following abstract version of Problem (P).

Problem ($\mathbf{P_0}$). Given $u_0 \in H$ and $\ell \in L^2(0, T; V^*)$, find $u \in L^2(0, T; V)$ such that

$$\begin{aligned} &\dot{u} + Lu + A(t)u + B(u, u) = \ell(t) \quad \text{a.e. in } (0, T) \\ &\text{and} \quad u(0) = u_0 \, . \end{aligned} \tag{26}$$

Note that if $u \in L^2(0, T; V)$, then $Lu + A(\cdot)u + B(u, u) \in L^1(0, T; V^*)$. Therefore, the initial-value problem (26) is meaningful and equivalent to an integral equation in the space V^*, namely,

$$u(t) = u_0 + \int_0^t (\ell(s) - Lu(s) - A(s)u(s) - B(u(s), u(s))) \, ds \quad \text{for all } t \in [0, T].$$

For this, it would of course suffice to assume that $u_0 \in V^*$ and $\ell \in L^1(0,T;V^*)$, but we will need the stronger assumptions of Problem (P_0) to obtain the a-priori estimates necessary to prove the following existence theorem.

Theorem. *Problem* (P_0) *has a solution* $u \in L^2(0,T;V)$, *which is weakly continuous as a mapping from* $[0,T]$ *into* H.

Before giving the proof, we note that the above theorem implies the global weak solvability of the original initial-boundary value problem, Problem (P). Indeed, if $u = (\hat{\mathbf{u}}, \hat{\mathbf{J}}) \in L^2(0,T;V)$ is a solution of Problem (P_0) with $u_0 = (\mathbf{u}_0 - \mathbf{u}^*(0), \mathbf{J}_0 - \mathbf{J}^*(0))$ and ℓ given by (24), then the functions $\mathbf{u} = \mathbf{u}^* + \hat{\mathbf{u}}$ and $\mathbf{J} = \mathbf{J}^* + \hat{\mathbf{J}}$ belong to $L^2(0,T;\mathbf{H}^1(\Omega))$ and $L^2(0,T;\mathbf{L}^2(\Omega))$, respectively; the divergence constraints (13), the boundary conditions (14), and the initial conditions (15) are all satisfied (in a weak sense, of course); and it is a matter of routine to construct functions p in $L^1(0,T;L^2(\Omega))$ and ϕ in $L^1(0,T;H^1(\Omega))$ such that the equations (11) and (12) hold in the sense of distributions. (In constructing p and ϕ, one simply employs some well-known properties of the gradient operator: it maps $L^2(\Omega)$ onto the polar set of V_1 in $\mathbf{H}^{-1}(\Omega)$ and maps $H^1(\Omega)$ onto the orthogonal complement of V_2 in $\mathbf{L}^2(\Omega)$; see, for example, [2, Chapter I.2.2].)

3. GALERKIN APPROXIMATION AND PROOF OF EXISTENCE

Since we have $V \hookrightarrow H = H^* \hookrightarrow V^*$, with compact dense embeddings, the isomorphism $L : V \to V^*$ induces a positive selfadjoint operator L_0 with compact inverse in the Hilbert space H. Let $(w_j)_{j\in\mathbf{N}}$ be a complete orthogonal sequence of eigenvectors of L_0. By definition of the operator, the sequence $(w_j)_{j\in\mathbf{N}}$ is complete and orthogonal in V and V^* as well.

For $n \in \mathbf{N}$ let $V_n = \mathrm{span}(w_1, \ldots, w_n)$ and let $P_n \in [H,H]$, $Q_n \in [V,V]$, and $Q_n^* \in [V^*,V^*]$ denote the orthogonal projections of H, V, and V^*, respectively, onto V_n. By our choice of the basis $(w_j)_{j\in\mathbf{N}}$, the projection Q_n is simply the restriction of P_n to V, and the projection Q_n^* is an extension of P_n to V^*; moreover, Q_n^* is the Banach space dual of Q_n.

We now consider the following finite-dimensional approximation of Problem (P_0).

Problem ($\mathbf{P_n}$). Given $u_0 \in H$ and $\ell \in L^2(0,T;V^*)$, find $u_n \in L^2(0,T;V_n)$ such that

$$\begin{aligned} &\dot{u}_n + Q_n^*(Lu_n + A(t)u_n + B(u_n,u_n)) = Q_n^*\,\ell(t) \quad \text{a.e. in } (0,T) \\ &\text{and} \quad u_n(0) = P_n u_0\,. \end{aligned} \tag{27}$$

Lemma 2. *For every* $n \in \mathbf{N}$, *Problem* (P_n) *has a unique solution* $u_n \in L^2(0,T;V_n)$. *The sequence* $(u_n)_{n\in\mathbf{N}}$ *is bounded in* $L^2(0,T;V)$ *and in* $L^\infty(0,T;H)$.

Proof. For every $n \in \mathbf{N}$, Problem (P_n) is simply an initial-value problem (in the sense of Carathéodory) for an ordinary differential equation in the finite-dimensional Banach space V_n. As such it has a (unique) maximal solution $u_n : I_n \to V_n$, defined and absolutely continuous on a relatively open subinterval I_n of $[0, T]$ with $0 \in I_n$. Since u_n satisfies the differential equation in (27) almost everywhere in I_n, we have

$$\begin{aligned}\langle \dot{u}_n, u_n \rangle_{V^*,V} &+ \langle L u_n, u_n \rangle_{V^*,V} \\ &+ \langle A(t) u_n, u_n \rangle_{V^*,V} + \langle B(u_n, u_n), u_n \rangle_{V^*,V} = \langle \ell(t), u_n \rangle_{V^*,V} \,,\end{aligned}$$

that is,

$$\langle \dot{u}_n, u_n \rangle_H + \langle u_n, u_n \rangle_V + a(t, u_n, u_n) + b(u_n, u_n, u_n) = \ell(t, u_n) \,.$$

Recalling the definitions (22) and (23) of the forms a and b and exploiting the antisymmetry property (20) of the form b_0, we see that $b(u_n, u_n, u_n) = 0$ and that $a(t, u_n, u_n) = b(u_n, u^*, u_n) = -b(u_n, u_n, u^*)$, where $u^* = (\mathbf{u}^*, \mathbf{J}^*)$. It follows that

$$\frac{1}{2} \frac{d}{dt} \|u_n\|_H^2 + \|u_n\|_V^2 = \ell(t, u_n) + b(u_n, u_n, u^*) \,.$$

From the discussion following the definition of b, we know that b is bounded (say, with norm β_∞) on $H \times V \times (L^\infty(\Omega))^2$. Thus, $b(u_n, u_n, u^*) \le \beta_\infty \|u^*\|_\infty \|u_n\|_H \|u_n\|_V$, where $\|u^*\|_\infty$ denotes the norm of $u^* = (\mathbf{u}^*, \mathbf{J}^*)$ in $(L^\infty((0,T) \times \Omega))^2$. We conclude that

$$\begin{aligned}\frac{d}{dt} \|u_n\|_H^2 + 2 \|u_n\|_V^2 &\le 2 \left(\|\ell(t)\|_{V^*} + \beta_\infty \|u^*\|_\infty \|u_n\|_H \right) \|u_n\|_V \\ &\le \left(\|\ell(t)\|_{V^*} + \beta_\infty \|u^*\|_\infty \|u_n\|_H \right)^2 + \|u_n\|_V^2\end{aligned}$$

and thus,

$$\frac{d}{dt} \|u_n\|_H^2 + \|u_n\|_V^2 \le 2 \left(\|\ell(t)\|_{V^*}^2 + \beta_\infty^2 \|u^*\|_\infty^2 \|u_n\|_H^2 \right) . \tag{28}$$

Dropping the second term on the left-hand side of (28) and integrating the inequality with respect to $t \in I_n$, we obtain

$$\begin{aligned}\|u_n(t)\|_H^2 &\le \|u_n(0)\|_H^2 + 2 \int_0^t \left(\|\ell(s)\|_{V^*}^2 + \beta_\infty^2 \|u^*\|_\infty^2 \|u_n(s)\|_H^2 \right) ds \\ &\le \|u_0\|_H^2 + 2 \|\ell\|_{L^2(0,T;V^*)}^2 + 2 \beta_\infty^2 \|u^*\|_\infty^2 \int_0^t \|u_n(s)\|_H^2 \, ds \,.\end{aligned}$$

Now Gronwall's lemma implies that

$$\|u_n(t)\|_H^2 \le \left(\|u_0\|_H^2 + 2 \|\ell\|_{L^2(0,T;V^*)}^2 \right) \exp(2 \beta_\infty^2 \|u^*\|_\infty^2 T) \,, \tag{29}$$

for all $t \in I_n$. In particular, $\|u_n\|_H$ is bounded on I_n, and this proves that $I_n = [0, T]$ and $u_n \in L^\infty(0, T; H)$. In fact, since the bound in (29) does not depend on n, the whole sequence $(u_n)_{n \in \mathbf{N}}$ is bounded in $L^\infty(0, T; H)$.

Next, we integrate (28) from 0 to T and obtain

$$\|u_n(T)\|_H^2 - \|u_n(0)\|_H^2 + \|u_n\|^2_{L^2(0,T;V)} \leq 2\,(\|\ell\|^2_{L^2(0,T;V^*)} + \beta_\infty^2 \|u^*\|_\infty^2 \|u_n\|^2_{L^2(0,T;H)})\,,$$

which implies

$$\|u_n\|^2_{L^2(0,T;V)} \leq \|u_0\|_H^2 + 2\,(\|\ell\|^2_{L^2(0,T;V^*)} + \beta_\infty^2 \|u^*\|_\infty^2 \|u_n\|^2_{L^\infty(0,T;H)} T)$$

and thereby, the boundedness of the sequence $(u_n)_{n\in\mathbf{N}}$ in $L^2(0,T;V)$.

Corollary. *The sequence $(u_n)_{n\in\mathbf{N}}$ of Lemma 2 contains a subsequence that converges, weakly in $L^2(0,T;V)$, strongly in $L^2(0,T;H)$, and weakly* in $L^\infty(0,T;H)$, to a function $u \in L^2(0,T;V)\cap L^\infty(0,T;H)$.*

Proof. Lemma 2 immediately implies the existence of a subsequence (u_{k_n}) of (u_n) that converges, weakly in $L^2(0,T;V)$ and weakly* in $L^\infty(0,T;H)$, to some function $u \in L^2(0,T;V)\cap L^\infty(0,T;H)$. Furthermore, we have

$$\dot{u}_n = Q_n^*(\ell(t) - Lu_n - A(t)u_n - B(u_n,u_n))\,,$$

almost everywhere in $(0,T)$, for $n \in \mathbf{N}$. Since (u_n) is bounded in $L^2(0,T;V)$ and the orthogonal projections Q_n^* are uniformly bounded in $[V^*,V^*]$, it follows that $(\dot{u}_n)$ is bounded in $L^1(0,T;V^*)$. But then a standard compactness argument (see, for example, [10, Theorem III.2.3]) applies and shows that $u_{k_n} \to u$ strongly in $L^2(0,T;H)$.

The above convergence results allow us to "pass to the limit" in Problem (P_n) and to prove that the limit point u of the sequence (u_n) is a solution of Problem (P_0). To argue the case, note that the initial-value problem (26) can be equivalently written in the following variational form:

$$\begin{aligned} -\langle u_0, v\rangle_H &- \int_0^T \langle u(t), v\rangle_H\, \dot{\varphi}(t)\,dt \\ &+ \int_0^T \langle u(t), v\rangle_V\, \varphi(t)\,dt + \int_0^T a(t,u(t),v)\,\varphi(t)\,dt \\ &+ \int_0^T b(u(t),u(t),v)\,\varphi(t)\,dt = \int_0^T \ell(t,v)\,\varphi(t)\,dt \end{aligned} \tag{30}$$

for all $v \in \mathcal{V}$ and all $\varphi \in C^\infty([0,T])$ with $\varphi(0) = 1$ and $\varphi(T) = 0$. Similarly, the initial-value problem (27) is equivalent to

$$\begin{aligned} -\langle P_n u_0, v\rangle_H &- \int_0^T \langle u_n(t), v\rangle_H\, \dot{\varphi}(t)\,dt \\ &+ \int_0^T \langle u_n(t), v\rangle_V\, \varphi(t)\,dt + \int_0^T a(t,u_n(t),Q_n v)\,\varphi(t)\,dt \\ &+ \int_0^T b(u_n(t),u_n(t),Q_n v)\,\varphi(t)\,dt = \int_0^T \ell(t,Q_n v)\,\varphi(t)\,dt \end{aligned} \tag{31}$$

for all $v \in \mathcal{V}$ and all $\varphi \in C^\infty([0,T])$ with $\varphi(0) = 1$ and $\varphi(T) = 0$. It thus suffices to show that each term in (31), with n replaced by k_n, converges to the corresponding term in (30) as $n \to \infty$.

The only term that requires some thought is, naturally, the one involving the trilinear form b. We write it as

$$\int_0^T b(u_n(t), u_n(t), Q_n v - v)\, \varphi(t)\, dt \; + \int_0^T b(u_n(t), u_n(t), v)\, \varphi(t)\, dt \tag{32}$$

and observe that

$$\int_0^T b(u_n(t), u_n(t), Q_n v - v)\, \varphi(t)\, dt \le \beta \, \|\varphi\|_{L^\infty(0,T)} \|u_n\|^2_{L^2(0,T;V)} \|Q_n v - v\|_V \,,$$

where β denotes the norm of b as a trilinear form on $V \times V \times V$. Since (u_n) is bounded in $L^2(0,T;V)$ and $\|Q_n v - v\|_V \to 0$ as $n \to \infty$, the first term in (32) vanishes in the limit. For the second term we have

$$\int_0^T b(u_n(t), u_n(t), v)\, \varphi(t)\, dt \le \beta_\infty \|\varphi\|_{L^\infty(0,T)} \|v\|_{(L^\infty(\Omega))^2} \|u_n\|_{L^2(0,T;H)} \|u_n\|_{L^2(0,T;V)} \,,$$

where β_∞ denotes, as in the proof of Lemma 2, the norm of b as a trilinear form on $H \times V \times (L^\infty(\Omega))^2$. Since $u_{k_n} \to u$, strongly in $L^2(0,T;H)$ and weakly in $L^2(0,T;V)$, it follows that

$$\int_0^T b(u_{k_n}(t), u_{k_n}(t), v)\, \varphi(t)\, dt \; \to \int_0^T b(u(t), u(t), v)\, \varphi(t)\, dt \,,$$

as desired.

Since the convergence of all other terms in Equation (31) is even easier to verify, we conclude that the limit point u of the sequence (u_n) is indeed a solution of Problem (P_0). As such, u is (absolutely) continuous as a mapping from $[0,T]$ into V^*. Also, we know already that u belongs to $L^\infty(0,T;H)$. But then a routine argument (see, for example, [10, Lemma III.1.4]) proves that u is in fact weakly continuous as a mapping from $[0,T]$ into H. That is, the initial condition $u(0) = u_0$ is satisfied in the sense of weak convergence in H (instead of just strong convergence in V^*).

This concludes the proof of the existence theorem stated at the end of Section 2 and thereby, the proof of global weak solvability of our original initial-boundary value problem, Problem (P).

ACKNOWLEDGEMENT

This material is based upon work supported by the National Science Foundation under Grant DMS-9625096.

REFERENCES

[1] R. Dautray and J.L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vols. 1–6, Springer, Berlin, 1988–93.

[2] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, Springer, New York, 1986.

[3] O.A. Ladyzhenskaya and V.A. Solonnikov, *Solutions of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid*, Trudy Mat. Inst. Steklov. 59 (1960), 115–173 (in Russian).

[4] O.A. Ladyzhenskaya and V.A. Solonnikov, *The linearization principle and invariant manifolds for problems of magnetohydrodynamics*, J. Soviet Math. 8 (1977), 384–422.

[5] A.J. Meir and P.G. Schmidt, *A velocity-current formulation for stationary MHD flow*, Appl. Math. Comp. 65 (1994), 95–109.

[6] A.J. Meir and P.G. Schmidt, *Variational methods for stationary MHD flow under natural interface conditions*, Nonlinear Analysis—Theory, Methods and Applications 26 (1996), 659–689.

[7] A.J. Meir and P.G. Schmidt, *Analysis and numerical approximation of a stationary MHD flow problem with nonideal boundary*, SIAM J. Numer. Anal. (to appear).

[8] J.A. Shercliff, *A Textbook of Magnetohydrodynamics*, Pergamon, Oxford, 1965.

[9] G. Ströhmer, *About the equations of non-stationary magneto-hydrodynamics with nonconducting boundaries*, Nonlinear Analysis—Theory, Methods and Applications (to appear).

[10] R. Temam, *Navier-Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984.